University of Alberta
Mathematical and Statistical Sciences
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Math 667, Topics in Differential Equations
Winter 2005

Assignment 1, due Jan 31. 2005, 9 AM in class

Exercise 1: (Linearization of discrete dynamical systems) (4)

We study the discrete dynamical system with differentiable function $f(x)$:

$$x_{n+1} = f(x_n).$$

1. Assume that $\bar{x}$ is a fixed point and consider small perturbations around $\bar{x}$ and define $x_n := \bar{x} + y_n$ where y_n is small. Derive the linearization of the above equation for y_n.
2. Prove the linear stability theorem which says that
If λ_j are the eigenvalues of $Df(\bar{x})$, and if $|\lambda_j| < 1$, then $\bar{x}$ is asymptotically stable.
(Hint: Use the 1-norm in $\mathbb{R}^n$ and show that $\|y_n\|_1 < \nu^n \|y_0\|_1$ for an appropriate constant $0 \le \nu < 1$.)

Exercise 2: (Spectral Theorem) (3)

Proof Theorem 1 in (1.3), which reads
(a) If μ is an eigenvalue of a real matrix A, then $\lambda = e^\mu$ is an eigenvalue of e^A.
(b) $Re\mu < 0$ if and only if $|\lambda| < 1$.

Exercise 3: (Stability of periodic orbits) (2)

Finish the proof of Theorem 1 in (1.6). Show that *if the periodic orbit γ is asymptotically stable for the flow $\phi_t(x)$, then x_0 is asymptotically stable for the Poincare-map P.*

Exercise 4: (Perko, p. 231, Problem Set 5, No. 1:) (5)

Show that the nonlinear system

$$\begin{aligned}\dot{x} &= -y + xz^2\\ \dot{y} &= x + yz^2\\ \dot{z} &= -z(x^2 + y^2)\end{aligned}$$

has a periodic orbit $\gamma(t) = (\cos t, \sin t, 0)$. Find the linearization of this system about $\gamma(t)$, the fundamental matrix $\Phi(t)$ for this autonomous linear system which satisfies $\Phi(0) = I$, and the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

Exercise 5: (6)

Use some computer software to solve the Lorenz-equations numerically:

$$\begin{aligned}\dot{x} &= 10(y - x)\\ \dot{y} &= \mu x - y - xz\\ \dot{z} &= xy - \frac{8}{3}z\end{aligned}$$

Here the Lorenz system is written in a form such that it depends only on one parameter μ (see Perko, p373 ff). Try different values of μ in the range of $0 < \mu < 30$ and investigate how the solutions look like. Generate the Lorenz Attractor which appears near $\mu = 28$.

We will arrange for a special meeting where everyone can present her or his solution.