Problem 1: [10]

1. Compute the Fourier series of the 2π-periodic function f given by

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2, \\ 0 & \text{if} \quad \pi/2 \le |x| \le \pi, \\ -1 & \text{if} \quad -\pi/2 < x < 0. \end{cases}$$

2. For which values of x does the Fourier series for f converge?

3. Sketch the graph of the function and of the Fourier series.

1. $a_0 = 0$

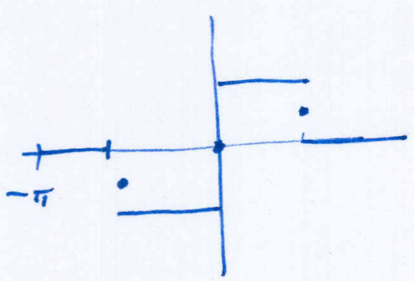
$a_n = 0$ f in odd 2

$b_n = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$ 2

$f(x) \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-\cos\frac{n\pi}{2}}{n} \sin nx$ 2

2. 2

3. 2



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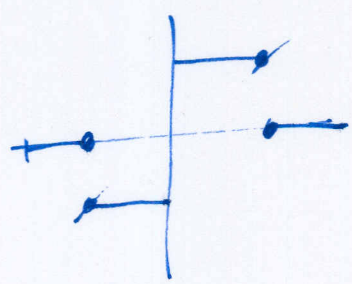


Problem 2: [15]

Consider the heat equation on $[0, \pi]$ with a source term

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + 3\sin x$$

subject to the boundary and initial conditions:

$$u(0,t) = 0,$$
$$u(\pi,t) = 0,$$
$$u(x,0) = 2\sin 4x.$$

1. Obtain the solution by the method of eigenfunction expansions.
2. Show that the solution approaches a steady-state solution as $t \to \infty$ and sketch the steady state solution.

1. eigenfunctions are $\phi_n(x) = \sin(nx)$ 2

$u = \sum a_n(t) \sin nx$

$q(x) = 3\sin x$ $\quad q_n = 0,\ q_1 = 3$ 3

$f(x) = 2\sin 4x$ $\quad f_n = 0,\ f_4 = 2$

$n \neq 1$: $\quad a_n'(t) = -kn^2 a_n \Rightarrow a_n(t) = a_n(0)e^{-kn^2 t}$

$n = 1$: $\quad a_1' = -ka_1 + 3 \Rightarrow a_1(t) = a_1(0)\, e^{-kt} + \frac{3}{k}(1 - e^{-kt})$

$n \neq 4$: $\quad a_n(0) = 0$

$n = 4$: $\quad a_4(0) = 2$

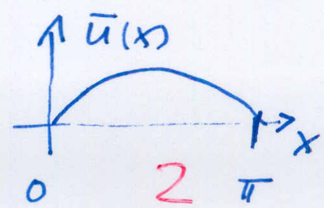
$\Rightarrow \quad a_n(t) = 0 \quad n \neq 1, 4$

$a_4(t) = 2e^{-16kt}$ $\quad a_1(t) = \frac{3}{k}(1 - e^{-kt})$ 6

Solution $\quad u(x,t) = \frac{3}{k}(1 - e^{-kt})\sin x + 2e^{-16kt}\sin 4x$

2. $\lim\limits_{t \to \infty} u(x,t) = \frac{3}{k}\sin x = \bar{u}(x)$

2



Problem 3: [10]

Solve the following first-order equation

$$\frac{\partial u}{\partial t} + 3x\frac{\partial u}{\partial x} = 2t, \quad -\infty < x < \infty, \quad t \geq 0$$

$$u(x,0) = \ln(1+x^2), \quad -\infty < x < \infty.$$

Characteristic equations

$$\frac{dx}{dt} = 3x \qquad X(t) = a e^{3t} \qquad a = x e^{-3t}$$ (3)

$$\frac{du}{dt} = 2t$$ (2) 5

$$u(x(t),t) = t^2 + K$$
$$= t^2 + u(a,0)$$
$$= t^2 + \ln(1+a^2)$$

5

$$u(x,t) = t^2 + \ln(1+x^2 e^{-6t})$$

Problem 4: [15]

Use the **energy method** to show that there are no negative eigenvalues for the Neumann problem

$$\frac{d^2\phi}{d\cancel{\phi^2}\ x} + \lambda\phi = 0, \quad 0 < x < L$$

$$\frac{d\phi}{dx}(0) = 0$$

$$\frac{d\phi}{dx}(L) = 0$$

This means, multiply the equation by ϕ, integrate and solve for λ. Does the expression for λ look familiar?

$$\int_0^L \phi \frac{d^2\phi}{dx^2}\, dx + \int_0^L \lambda \phi^2 dx = 0$$

$$\underbrace{\phi \frac{d\phi}{dx}\Big|_0^L}_{=0} - \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx + \lambda \int_0^L \phi^2 dx = 0$$ 8

$$\lambda = \frac{\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 dx} \geq 0$$ 5

This is the Rayleigh quotient. 2