# Math 337, Summer 2010
# Assignment 3

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**Exercise 0.1.** ✗
Find the values of $\lambda^2$ for which the boundary value problem

$$\frac{d^2u}{dx^2} + \lambda^2 u = 0, \quad 0 < x < \frac{\pi}{2}$$
$$u(0) = 0$$
$$\int_0^{\frac{\pi}{2}} u(t)\,dt = 0$$

has nontrivial solutions.

**Solution to Exercise 0.1:** We consider two cases:
case (i): $\lambda = 0$
In this case, the general solution to $\dfrac{d^2u}{dx^2} = 0$ is given by $u(x) = Ax + B$, and $u(0) = 0$ implies that $B = 0$, so that $u(x) = Ax$.
The condition $\int_0^{\frac{\pi}{2}} u(t)\,dt = 0$ implies that

$$\int_0^{\frac{\pi}{2}} A\,t\,dt = A\,\frac{t^2}{2}\bigg|_0^{\frac{\pi}{2}} = A\,\frac{\pi^2}{8} = 0,$$

which implies that $A = 0$, and the boundary value problem has only the trivial solution in this case.
case (ii): $\lambda \neq 0$
In this case, the general solution to $\dfrac{d^2u}{dx^2} + \lambda^2 u = 0$ is given by $u(x) = A\cos\lambda x + B\sin\lambda x$, and $u(0) = 0$ implies that $A = 0$ so that $u(x) = B\sin\lambda x$.
The condition $\int_0^{\frac{\pi}{2}} u(t)\,dt = 0$ implies that

$$\int_0^{\frac{\pi}{2}} B\sin\lambda t\,dt = -\frac{B}{\lambda}\cos\lambda t\bigg|_0^{\frac{\pi}{2}} = \frac{B}{\lambda}\left(1 - \cos\frac{\lambda\pi}{2}\right) = 0,$$

and so either $B = 0$ or $\cos\frac{\lambda\pi}{2} = 1$.

Therefore, a nontrivial solution exists if and only if we have $\cos\frac{\lambda\pi}{2} = 1$, that is, $\frac{\lambda\pi}{2} = 2\pi n$, where $n \neq 0$ is an integer. The values of $\lambda^2$ for which the boundary value problem has non-trivial solutions are

$$\lambda_n^2 = 16n^2,$$

for $n = 1, 2, 3, \ldots$.

**Exercise 0.2.** ✗✗
Consider the following eigenvalue problem on the interval $[\,0, 1\,]$ :

$$\begin{aligned} u''(x) + 2u'(x) - u(x) + \lambda\,(x+1)^2 e^{-2x}\,u(x) &= 0 \\ u'(0) &= 0 \\ u'(1) &= 0 \end{aligned}$$

(a) Explain the meaning of *eigenvalue problem.*

(b) Show that this eigenvalue problem is not of Sturm-Liouville type.

(c) Multiply the above equation by $e^{2x}$ to obtain a Sturm-Liouville problem. Identify $p(x)$, $q(x)$, and $\sigma(x)$.

(d) Use the Rayleigh quotient to show that the leading eigenvalue is positive, that is, $\lambda_1 > 0$.

(e) Find an upper bound for the leading eigenvalue.

**Solution to Exercise 0.2:**

(a) The *eigenvalue problem* consists of finding the values of $\lambda$ (*eigenvalues*) for which there are nontrivial solutions (*eigenfunctions*) satisfying both the differential equation and the boundary conditions.

(b) If the eigenvalue problem

$$\begin{aligned} u''(x) + 2u'(x) - u(x) + \lambda\,(x+1)^2 e^{-2x}\,u(x) &= 0 \\ u'(0) &= 0 \\ u'(1) &= 0 \end{aligned}$$

were of Sturm-Liouville form, then we would have

$$u'' + 2u' = (p\,u')',$$

that is, $p(x) = 1$ and $p'(x) = 2$, which is impossible.

(c) If we multiply the differential equation by $e^{2x}$, then we have

$$e^{2x}u'' + 2e^{2x}u' - e^{2x}u + \lambda\,(x+1)^2 u = 0, \quad 0 \leqslant x \leqslant 1,$$

that is,

$$(e^{2x}u')' - e^{2x}u + \lambda(x+1)^2 u = 0, \quad 0 \leqslant x \leqslant 1,$$

which is of Sturm-Liouville type with $p(x) = e^{2x},\ q(x) = -e^{2x},\ \sigma(x) = (x+1)^2$ for $0 \leqslant x \leqslant 1$.

(d) The eigenvalue $\lambda$ and corresponding eigenfunction $u$ are related by the Rayleigh quotient:

$$\lambda = R(u) = \frac{-p(x)u(x)u'(x)\Big|_0^1 + \displaystyle\int_0^1 \left[p(x)u'(x)^2 - q(x)u(x)^2\right]\,dx}{\displaystyle\int_0^1 u(x)^2\sigma(x)\,dx},$$

and

$$-p(x)u(x)u'(x)\Big|_0^1 = -e^2u(1)u'(1) + u(0)u'(0) = 0,$$

since $u'(0) = 0$ and $u'(1) = 0$, so the Rayleigh quotient becomes

$$\lambda = R(u) = \frac{\displaystyle\int_0^1 \left[e^{2x}u'(x)^2 + e^{2x}u(x)^2\right]\,dx}{\displaystyle\int_0^1 u(x)^2(x+1)^2\,dx} \geqslant 0,$$

and all the eigenvalues of the boundary value problem are nonnegative.

In order to show that $\lambda = 0$ is not an eigenvalue, we can see immediately that since $u$ and $u'$ are continuous on the interval $[0,1]$, then

$$\int_0^1 \left[e^{2x}u'(x)^2 + e^{2x}u(x)^2\right]\,dx = 0,$$

implies that $u(x) = u'(x) = 0$ for all $0 \leqslant x \leqslant 1$. Hence there is no nontrivial eigenfunction corresponding to $\lambda = 0$.

Alternatively, the equation $u'' + 2u' - u = 0,\ 0 \leqslant x \leqslant 1$, has general solution

$$u(x) = Ae^{-x}e^{\sqrt{2}x} + Be^{-x}e^{-\sqrt{2}x},$$

and from the boundary conditions $u'(0) = 0$ and $u'(1) = 0$, the solution is $u(x) = 0$ for $0 \leqslant x \leqslant 1$.

Therefore the leading eigenvalue $\lambda_1 > 0$.

(e) In order to get an upper bound on $\lambda_1$, we try a quadratic test function $v$ which satisfies the boundary conditions $v'(0) = 0$ and $v'(1) = 0$, say

$$v(x) = ax^2 + bx + c \qquad \text{with} \qquad v'(x) = 2ax + b,$$

then the boundary conditions imply that $a = b = 0$, so that $v(x) = c$ for $0 \leqslant x \leqslant 1$.

The Rayleigh quotient for this test function is

$$R(v) = \frac{\displaystyle\int_0^1 c^2 e^{2x}\, dx}{\displaystyle\int_0^1 c^2 (x+1)^2\, dx} = \frac{3}{14}(e^2 - 1).$$

and since $\lambda_1$ is the minimum of $R(u)$ as $u$ runs over all twice continuously differentiable functions that satisfy the boundary conditions, then

$$0 < \lambda_1 \leqslant \frac{3}{14}(e^2 - 1).$$

**Exercise 0.3.** ✗✗

Hermite's differential equation reads

$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty$$

(a) Multiply by $e^{-x^2}$ and bring the differential equation into Sturm-Liouville form. Decide if the resulting Sturm-Liouville problem is regular or singular.

(b) Show that the Hermite polynomials

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \qquad H_3(x) = 8x^3 - 12x$$

are eigenfunctions of the Sturm-Liouville problem and find the corresponding eigenvalues.

(c) Use an appropriate weight function and show that $H_1$ and $H_2$ are orthogonal on the interval $(-\infty, \infty)$ with respect to this weight function.

**Solution to Exercise 0.3:**

(a) Multiplying the differential equation by $e^{-x^2}$, we have

$$e^{-x^2} y'' - 2xe^{-x^2} y' + \lambda e^{-x^2} y = 0,$$

that is,

$$\frac{d}{dx}\left(e^{-x^2}y'\right) + \lambda e^{-x^2}y = 0.$$

This is the self-adjoint form of Hermite's equation, with $p(x) = r(x) = e^{-x^2}$ and $q(x) = 0$. Even though $p,\ p',\ q,$ and $r$ are all continuous on the interval $(-\infty, \infty)$, this Sturm-Liouville problem is singular since the interval is infinite.

(b) The Hermite polynomial of degree $n$ is denoted by $H_n(x)$.

- For $H_0(x) = 1$ we have
$$H_0'' - 2xH_0' + \lambda_0 H_0 = 0$$
if and only if $\lambda_0 = 0$, and the eigenvalue corresponding to the eigenfunction $H_0(x)$ is $\lambda_0 = 0$.
- For $H_1(x) = 2x$ we have
$$H_1'' - 2xH_1' + \lambda_1 H_1 = 0$$
if and only if $-4x + 2\lambda_1 x = 0$ for all $x$, that is, if and only if $\lambda_1 = 2$, and the eigenvalue corresponding to the eigenfunction $H_1(x)$ is $\lambda_1 = 2$.
- For $H_2(x) = 4x^2 - 2$ we have
$$\begin{aligned} H_2'' - 2xH_2' + \lambda_2 H_2 &= 8 - 2x(8x) + \lambda_2(4x^2 - 2) \\ &= -4(4x^2 - 2) + \lambda_2(4x^2 - 2) \\ &= (\lambda_2 - 4)(4x^2 - 2) \\ &= 0 \end{aligned}$$
for all $x$ if and only if $\lambda_2 = 4$, and the eigenvalue corresponding to the eigenfunction $H_2(x)$ is $\lambda_2 = 4$.
- For $H_3(x) = 8x^3 - 12x$ we have
$$\begin{aligned} H_3'' - 2xH_3' + \lambda_3 H_3 &= 48x - 2x(24x^2 - 12) + \lambda_3(8x^3 - 12x) \\ &= -48x^3 + 72x + \lambda_3(8x^3 - 12x) \\ &= (\lambda_3 - 6)(8x^3 - 12x) \\ &= 0 \end{aligned}$$
for all $x$ if and only if $\lambda_3 = 6$, and the eigenvalue corresponding to the eigenfunction $H_3(x)$ is $\lambda_3 = 6$.

(c) There are two methods to answer this question. The more elegant method is as follows. We can show that the Hermite polynomials $H_n$, for $n \geqslant 0$, are orthogonal on the interval $(-\infty, \infty)$ with respect to the weight function $r(x) = e^{-x^2}$, by noting that

$$\begin{aligned} &e^{-x^2}H_m H_n'' - 2xe^{-x^2}H_m H_n' + \lambda_n H_m H_n = 0, \quad \text{and} \\ &e^{-x^2}H_n H_m'' - 2xe^{-x^2}H_n H_m' + \lambda_m H_m H_n = 0, \end{aligned}$$

and subtracting, we have

$$\frac{d}{dx}\left[e^{-x^2}\left(H_m H_n' - H_n H_m'\right)\right] + (\lambda_n - \lambda_m)e^{-x^2} H_m H_n = 0.$$

integrating over the real line, we have

$$(\lambda_n - \lambda_m)\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = \lim_{M\to\infty} e^{-x^2}\left[H_n(x)H_m'(x) - H_m(x)H_n'(x)\right]\Big|_{-M}^{M} = 0$$

since the exponential kills off any polynomial as $|x| \to \infty$. Therefore, if $m \neq n$, then

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = 0.$$

A more straightforward method is by integrating directly, we note immediately that

$$\int_{-\infty}^{\infty} e^{-x^2} H_1(x)H_2(x)\,dx = 0,$$

since the integrand is an odd function of $x$ and we are integrating between symmetric limits.

**Exercise 0.4.** ✗

Consider torsional oscillations of a homogeneous cylindrical shaft. If $\omega(x,t)$ is the angular displacement at time $t$ of the cross section at $x$, then

$$\frac{\partial^2 \omega}{\partial t^2} = a^2 \frac{\partial^2 \omega}{\partial x^2}, \qquad 0 \leqslant x \leqslant L, \quad t > 0.$$

where the initial conditions are

$$\omega(x,0) = f(x), \qquad \text{and} \qquad \frac{\partial \omega}{\partial t}(x,0) = 0, \quad 0 \leqslant x \leqslant L,$$

and the ends of the shaft are fixed elastically:

$$\frac{\partial \omega}{\partial x}(0,t) - \alpha\,\omega(0,t) = 0, \qquad \text{and} \qquad \frac{\partial \omega}{\partial x}(L,t) + \alpha\,\omega(L,t) = 0, \quad t > 0$$

with $\alpha$ a positive constant.

(a) Why is it possible to use separation of variables to solve this problem?

(b) Use separation of variables and show that one of the resulting problems is a regular Sturm-Liouville problem.

(c) Show that all of the eigenvalues of this regular Sturm-Liouville problem are positive.

**Note:** You do not need to solve the initial value problem, just answer the questions (a), (b), and (c).

**Solution of Exercise 0.4:**

(a) Since the partial differential equation is linear and homogeneous and the boundary conditions are linear and homogeneous, we can use separation of variables.

(b) Assuming a solution of the form

$$\omega(x,t) = \phi(x)\cdot G(t), \quad 0 \leqslant x \leqslant L, \quad t \geqslant 0$$

and separating variables, we have two ordinary differential equations:

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0, \quad 0 \leqslant x \leqslant L, \qquad G''(t) + \lambda a^2 G(t) = 0, \quad t > 0, \\ \phi'(0) - \alpha\phi(0) &= 0 \\ \phi'(L) + \alpha\phi(L) &= 0 \end{aligned}$$

where the $\phi$-problem is a regular Sturm-Liouville problem with

$$p(x) = 1, \qquad q(x) = 0, \qquad r(x) = 1$$

and

$$\beta_1 = 1, \qquad \beta_2 = -\alpha, \qquad \beta_3 = 1, \qquad \beta_4 = \alpha.$$

(c) We use the Rayleigh quotient to show that $\lambda > 0$ for all eigenvalues $\lambda$.

Let $\lambda$ be an eigenvalue of the Sturm-Liouville problem, and let $\phi(x)$ be the corresponding eigenfunction, then

$$-p(x)\phi(x)\phi'(x)\Big|_0^L = -\phi(L)\phi'(L) + \phi(0)\phi'(0) = \alpha(\phi(0)^2 + \phi(L)^2) > 0,$$

and since $q(x) = 0$ for all $0 \leqslant x \leqslant L$, then

$$\lambda = \frac{\alpha(\phi(0)^2 + \phi(L)^2) + \displaystyle\int_0^L \phi'(x)^2\,dx}{\displaystyle\int_0^L \phi(x)^2\,dx} \geqslant 0$$

since $p(x) = \sigma(x) = 1$ for $0 \leqslant x \leqslant L$.

Note that if $\lambda = 0$, then

$$\alpha\big(\phi(0)^2 + \phi(L)^2\big) + \int_0^L \phi'(x)^2\,dx = 0$$

implies that

$$\alpha\big(\phi(0)^2 + \phi(L)^2\big) = 0 \qquad \text{and} \qquad \int_0^L \phi'(x)^2\,dx = 0.$$

Since $\alpha > 0$, this implies that $\phi(0) = 0$ and $\phi(L) = 0$; and since $\phi'$ is continuous on $[0, L]$, that $\phi'(x) = 0$ for $0 \leqslant x \leqslant L$. Therefore $\phi(x)$ is constant on $[0, L]$, so that $\phi(x) = \phi(0) = 0$ for $0 \leqslant x \leqslant L$, and $\lambda = 0$ is not an eigenvalue. Thus, all of the eigenvalues $\lambda$ of this Sturm-Liouville problem satisfy $\lambda > 0$.

**Exercise 0.5.** ✗

Solve the following initial value problem for the damped wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= \frac{1}{1+x^2}, \quad -\infty < x < \infty \\ \frac{\partial u}{\partial t}(x,0) &= 1, \quad -\infty < x < \infty. \end{aligned}$$

**Hint:** Do not use separation of variables, instead solve the initial value – boundary value problem satisfied by $w(x,t) = e^t \cdot u(x,t)$.

**Solution to Exercise 0.5:** Note that $u(x,t) = e^{-t} \cdot w(x,t)$, so that

$$\frac{\partial^2 u}{\partial x^2} = e^{-t}\frac{\partial^2 w}{\partial x^2}$$

and

$$\frac{\partial u}{\partial t} = -e^{-t}w + e^{-t}\frac{\partial w}{\partial t}$$

and

$$\frac{\partial^2 u}{\partial t^2} = e^{-t}w - 2e^{-t}\frac{\partial w}{\partial t} + e^{-t}\frac{\partial^2 w}{\partial t^2}.$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = e^{-t}\frac{\partial^2 w}{\partial t^2},$$

while

$$\frac{\partial^2 u}{\partial x^2} = e^{-t}\frac{\partial^2 w}{\partial x^2}$$

and if $u$ is a solution to the original partial differential equation, then $w$ is a solution to the equation

$$e^{-t}\left[\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2}\right] = 0,$$

and since $e^{-t} \neq 0$, then $w$ satisfies the initial value problem

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \frac{\partial^2 w}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \\ w(x,0) &= \frac{1}{1+x^2}, \\ \frac{\partial w}{\partial t}(x,0) &= 1 + \frac{1}{1+x^2}. \end{aligned}$$

From d'Alembert's solution to the wave equation, we have (since $c = 1$)

$$w(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t}\left(1 + \frac{1}{1+s^2}\right) ds,$$

so that

$$u(x,t) = \frac{e^{-t}}{2}\left[\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2}\right] + \frac{e^{-t}}{2}\left[2t + \tan^{-1}(x+t) - \tan^{-1}(x-t)\right],$$

for $-\infty < x < \infty, \quad t \geqslant 0.$