Math 337, Summer 2010
Assignment 1

Dr. T Hillen, University of Alberta

Exercise 0.1. ✗

Let $v(x)$ be the steady-state solution to the initial boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + r = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0$$

$$u(0,t) = T_0, \quad t > 0$$

$$\frac{\partial u}{\partial x}(a,t) = 0, \quad t > 0$$

where r is a constant. Find and solve the boundary value problem for the steady-state solution $v(x)$.

Solution to Exercise 0.1: The steady-state solution $v(x)$ satisfies the boundary value problem

$$\frac{d^2 v}{dx^2} + r = 0, \quad 0 < x < a$$

$$v(0) = T_0$$

$$\frac{dv}{dx}(a) = 0,$$

and the general solution to the differential equation is

$$v(x) = -\frac{1}{2}rx^2 + Ax + B,$$

and

$$\frac{dv}{dx}(x) = -rx + A.$$

Therefore,

$$v(0) = T_0 \quad \text{implies} \quad B = T_0$$

$$\frac{dv}{dx}(a) = 0 \quad \text{implies} \quad -ra + A = 0,$$

so that

$$A = ra \qquad \text{and} \qquad B = T_0.$$

The steady-state solution is therefore

$$v(x) = -\frac{1}{2}rx^2 + rax + T_0$$

for $0 \leqslant x \leqslant a$.

Exercise 0.2. ✗
Solve the normalized wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant \pi, \quad t \geqslant 0$$

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \geqslant 0$$

$$u(x,0) = \sin x, \quad \frac{\partial u}{\partial t}(x,0) = \sin x, \quad 0 \leqslant x \leqslant \pi.$$

Solution to Exercise 0.2: Since the partial differential equation and the boundary conditions are linear and homogeneous, we can use separation of variables, and assuming a solution of the form $u(x,t) = \phi(x) \cdot G(t)$, we get two ordinary differential equations:

$$\begin{aligned} \phi'' + \lambda\phi = 0, \quad 0 \leqslant x \leqslant \pi \qquad & G'' + \lambda G = 0, \quad t \geqslant 0 \\ \phi(0) = 0, & \\ \phi(\pi) = 0, & \end{aligned}$$

where λ is the separation constant.
We solve the spatial problem first since it has a complete set of boundary conditions. These are homogeneous Dirichlet conditions, so the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2 \qquad \text{and} \qquad \phi_n(x) = \sin nx$$

for $n \geqslant 1$, and the corresponding solutions to the temporal equation are

$$G_n(t) = a_n \cos nt + b_n \sin nt.$$

Using the superposition principle we write the solution as an "infinite" linear combination of $\{\phi_n \cdot G_n\}_{n \geqslant 1}$, that is,

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left(a_n \cos nt + b_n \sin nt\right),$$

where the constants a_n and b_n are determined from the initial conditions

$$\sin x = u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx$$

and
$$\sin x = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n b_n \sin nx.$$
From the orthogonality of the eigenfunctions, we find
$$\begin{aligned} a_1 &= 1, \quad a_k = 0 \quad \text{for} \quad k \neq 1 \\ b_1 &= 1, \quad b_k = 0 \quad \text{for} \quad k \neq 1 \end{aligned}$$
and the solution is
$$u(x,t) = \sin x \cos t + \sin x \sin t$$
for $0 \leqslant x \leqslant \pi$, and $t \geqslant 0$.

Exercise 0.3. ✗
Find all functions ϕ for which $u(x,t) = \phi(x - ct)$ is a solution of the heat equation
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad -\infty < x < \infty$$
where k and c are constants.

Solution to Exercise 0.3: If ϕ is a twice continuously differentiable function such that $u(x,t) = \phi(x - ct)$ is a solution of the heat equation, then
$$\begin{aligned} \frac{\partial u}{\partial t} &= -c\phi'(x - ct) \\ \frac{\partial u}{\partial x} &= \phi'(x - ct) \\ \frac{\partial^2 u}{\partial x^2} &= \phi''(x - ct) \end{aligned}$$
and ϕ satisfies the equation
$$\phi''(x - ct) + \frac{c}{k}\phi'(x - ct) = 0,$$
for all $-\infty < x < \infty$ and $t \geqslant 0$, that is,
$$\phi''(s) + \frac{c}{k}\phi'(s) = 0$$
for all $s \in \mathbb{R}$. Therefore the solution is given by
$$\phi(s) = A + Be^{-\frac{c}{k}s},$$
that is,
$$u(x,t) = A + Be^{-\frac{c}{k}(x-ct)}$$
where A and B are arbitrary constants.

Exercise 0.4. ✗

A fluid occupies the half plane $y > 0$ and flows past (left to right, approximately) a plate located near the x-axis. If the x and y components of the velocity are $U_0 + u(x, y)$ and $v(x, y)$, respectively where U_0 is the constant free-stream velocity, then under certain assumptions, the equations of motion, continuity, and state can be reduced to

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad \left(1 - M^2\right)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{$*$}$$

valid for all $-\infty < x < \infty,\ 0 < y < \infty$.
Suppose there exists a function ϕ (called the *velocity potential*), such that

$$u = \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad v = \frac{\partial \phi}{\partial y}.$$

(a) State a condition under which the first equation in $(*)$ above becomes an identity.

(b) Show that the second equation in $(*)$ above becomes (assuming the free-stream Mach number M is a constant) a partial differential equation for ϕ which is elliptic if $M < 1$ or hyperbolic if $M > 1$.

Solution to Exercise 0.4:

(a) If the velocity potential ϕ exists, then

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x}$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x \partial y},$$

and the mixed partial derivatives are equal at all points where they are continuous. Therefore, the first equation in $(*)$ is an identity provided that

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

for all $-\infty < x < \infty,\ 0 < y < \infty$.

Another possible solution then is obtained by assuming that the velocity potential $\phi(x, y)$ is twice continuously differentiable.

(b) Again, assuming the existence of a velocity potential, the second equation in $(*)$ becomes

$$\left(1 - M^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

which is elliptic if $1 - M^2 > 0$ and hyperbolic if $1 - M^2 < 0$, that is, elliptic if $M < 1$ and hyperbolic if $M > 1$.

Solution to Exercise 0.4:

(a) If the velocity potential ϕ exists, then

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x}$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x \partial y},$$

and the mixed partial derivatives are equal at all points where they are continuous. Therefore, the first equation in $(*)$ is an identity provided that

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

for all $-\infty < x < \infty,\ 0 < y < \infty$.

Another possible solution then is obtained by assuming that the velocity potential $\phi(x, y)$ is twice continuously differentiable.

(b) Again, assuming the existence of a velocity potential, the second equation in $(*)$ becomes

$$\left(1 - M^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

which is elliptic if $1 - M^2 > 0$ and hyperbolic if $1 - M^2 < 0$, that is, elliptic if $M < 1$ and hyperbolic if $M > 1$.

Solution to Exercise 0.4:

(a) If the velocity potential ϕ exists, then

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x}$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x \partial y},$$

and the mixed partial derivatives are equal at all points where they are continuous. Therefore, the first equation in $(*)$ is an identity provided that

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

for all $-\infty < x < \infty,\ 0 < y < \infty$.

Another possible solution then is obtained by assuming that the velocity potential $\phi(x, y)$ is twice continuously differentiable.

(b) Again, assuming the existence of a velocity potential, the second equation in $(*)$ becomes

$$\left(1 - M^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

which is elliptic if $1 - M^2 > 0$ and hyperbolic if $1 - M^2 < 0$, that is, elliptic if $M < 1$ and hyperbolic if $M > 1$.