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The one-dimensional chemotaxis model: global existence and asymptotic profile

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SUMMARY

Osaki and Yagi (2001) give a proof of global existence for the classical chemotaxis model in one space dimension with use of energy estimates. Here we present an alternative proof which uses the regularity properties of the heat-equation semigroup. With this method we can identify a large selection of admissible spaces, such that the chemotaxis model defines a global semigroup on these spaces.

We use scaling arguments to derive the asymptotic profile of the solutions and we show numerical simulations. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: chemotaxis; global existence; advection–diffusion equations

Dedicated to Professor Howard Levine, a great scientist, teacher, and friend.

1. INTRODUCTION

In this paper we study local and global existence for the following one-dimensional chemotaxis model:

$$\begin{aligned} u_t &= u_{xx} - \chi(uv_x)_x \\ v_t &= \varepsilon v_{xx} + u - av \end{aligned} \tag{1}$$

on a bounded interval $\Omega = [0, l]$ with homogeneous Neumann boundary conditions

$$u_x(t,0) = u_x(t,l) = v_x(t,0) = v_x(t,l) = 0 \tag{2}$$

or with periodic boundary conditions

$$u(t,0) = u(t,l), \quad u_x(t,0) = u_x(t,l), \quad v(t,0) = v(t,l), \quad v_x(t,0) = v_x(t,l) \tag{3}$$

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The parameters a, d, and χ are constants. We prove that solutions are global in time for all $\varepsilon > 0$ and we study the case of $\varepsilon \to 0$ qualitatively and numerically.

The above model appears as a special case of the Patlak-Keller-Segel equations for chemotaxis [1,2]. The model has been used to describe the oriented movement of cell populations, guided by a chemical gradient which is produced by the cells themselves. In the biological interpretation $u(x,t)$ is a (scalar) particle density, $v(x,t)$ denotes the chemical concentration, χ is called chemotactic sensitivity, ε is the diffusion constant for the signal, the diffusion constant of the species has been scaled to 1, and a is the decay rate of the chemical signal.

Model (1) is easily extended to more than one space dimension (just replace the space derivative by ∇). And it is known that solutions to (1) in dimensions $n \geqslant 2$ can blow up in finite time. For $n = 2$ there is a threshold such that blow-up happens, whereas for $n \geqslant 3$ blow-up can happen for small initial particle densities (e.g. References [3–8]). Model (1) was already studied by Childress and Percus [9] from a theoretical point of view. They use scaling arguments which speak against finite time blow-up in one dimension. Nagai [10] announced a result on equations (1) with no diffusion in the signal $\varepsilon = 0$, which states that the solution (u, v) blows up in finite or infinite time. Levine and Sleeman [11] constructed an explicit solution to (1) with $\varepsilon = 0$ and $a = 0$ which blows up in finite time. Numerical simulations (not shown here) suggest that system (1) with $\varepsilon = 0$ and $a \neq 0$ should have finite time blow-up solutions as well, but a proof of that is still missing.

For $\varepsilon > 0$ solutions exist globally as was recently shown by Osaki and Yagi [12]. They studied solutions in Hilbert spaces, $(u, v) \in L^2 \times H^1$ and they use energy estimates to prove global existence. Moreover, they show the existence of a global attractor under certain assumptions. Osaki and Yagi allow more general chemotactic sensitivities $\chi(v)$, but they need two basic assumptions, which are not needed in the proof presented here. First they study solutions in separable spaces, in particular L^2 and H^1, since energy methods are used. In this paper, we do not assume separability and we give a large selection of admissible spaces, such that system (1) defines a strongly continuous global solution semigroup on those spaces. We give a systematic way to find admissible spaces in Appendix A. Our proof of global existence is based on the regularity properties of the heat-equation semigroup (see Lemma 2.1). It presents a very transparent way to deal with cross diffusion systems in general and this method was successfully applied in Hillen and Painter [13] to a related problem. We can also see in our proof that the result cannot be generalized to two or higher dimensions. This makes sense, since we know about finite time bow-up in higher dimensions. A second difference to Osaki and Yagi appears related to the initial conditions. Osaki and Yagi assume throughout their paper that the initial condition for v is bounded away from 0 : $\inf_\Omega v_0(x) \geqslant \delta > 0$. An assumption of that nature is not needed here, we only use $v_0 \geqslant 0$. Osaki and Yagi show the existence of an attractor, we derive the asymptotic profile for u and v as $t \to \infty$.

The semigroup approach of this paper was also used in Yagi [14] for the two-dimensional chemotaxis model. There it is mentioned that the semigroup approach is applicable to the one-dimensional case as well. This was not done in the paper of Yagi, so we do it here. In doing so it turns out that the application to one dimension is not so straightforward. In particular, some of the estimates which we use in our proof are not extendible to two dimensions. Also Yagi uses the assumption of $\inf_\Omega v_0(x) \geqslant \delta > 0$, which we can relax here.

In this paper we first identify appropriate Sobolev spaces, such that solutions to (1) exist locally. We find that a choice of $(u, v) \in L^\infty \times W^{\sigma,p}$ is appropriate for $1 < \sigma < 2$ and

$p > 1/(\sigma - 1)$. We extend the *a priori* estimate of Childress and Percus to global estimates for the solutions constructed above. This in fact proves global existence in time and excludes any form of blow up.

We use scaling arguments and numerical simulation to show that the maximum of the solution grows like $1/\varepsilon$ for $\varepsilon \to 0$. In numerical examples we show how the solutions grow fast to a level of order $1/\varepsilon$ until they saturate.

The proofs for local and global existence use a combination of an L^p space for u and a space of higher regularity, $W^{\sigma,p}$, for v with $1 < \sigma < 2$. Such spaces were applied to related problems in Yagi [14], Hillen and Painter [13], and Osaki *et al.* [15]. In Hillen and Painter a modified chemotaxis model is studied which accounts for volume filling effects of particles which have finite volume. There local and global existence is proven for each space dimension $n \geqslant 1$. In Osaki *et al.* the classical chemotaxis model is studied with the addition of growth and death terms for the cellular species. The existence of an exponential attractor is proven in $L^2 \times H^{1+\varepsilon_0}$. In all five articles, [12–15] and this one, it turns out that, by choosing a space of higher regularity for v, we take advantage of the triangular form of the leading order differential operator in (1). If the system would have cross-diffusion terms in both equations, then this method would not work. In that case, the theory of Amann and others applies [16] and we have to ensure that the corresponding coefficient matrix is coercive. Here we do not need that. Since v has higher regularity, the v_{xx}-term appears as a lower order term, compared to u_{xx}.

The paper is organized as follows. In Section 2 we state the existence results. In Section 3 we prove local existence in the space $L^\infty \times W^{\sigma,p}$, $1 < \sigma < 2, p > n/(\sigma - 1)$ for each space dimension $n \geqslant 1$. The proof uses Banach's fixed-point theorem. In Section 4 we prove global existence in time for the 1-D case and we see that this proof cannot be extended to space dimensions $n \geqslant 2$. The proof starts from the global L^1 estimate for u (particle conservation). Then we use regularity properties of the solution semigroup of the heat equation to derive global $W^{\sigma,r}$ estimates for v, for $r < p$. From these we find L^p estimates for u, with $p > 1$, for example $p = 2$ would work. These estimates in turn provide us with $W^{\sigma,p}$ estimates for v which then are used to show L^∞ estimates for u. This proves global existence. In Section 5 we show some numerical simulations and we use singular perturbation methods to find asymptotic profiles for the aggregation peaks.

2. STATEMENT OF THE RESULTS

The whole analysis of this paper relies on the following lemma about the regularity of the solution semigroup of the heat equation, as taken from Taylor [17]:

Lemma 2.1 (*Taylor* [*17, p. 274*])
Let M be a bounded n-dimensional C^∞ manifold without boundary. Let $T_1(t) = e^{\Delta t}$ denote the solution semigroup of the heat equation on M. Assume

$$0 < t \leqslant 1, \quad p \geqslant q, \quad s \geqslant r \tag{4}$$

then

$$T_1(t) : W^{r,q}(M) \to W^{s,p}(M) \quad \text{with norm } Ct^{-\alpha} \tag{5}$$

where

$$-\alpha = -\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{2}(s-r)$$

The assumptions on M are met by the heat equation on a rectangular $[0, l_1] \times \cdots \times [0, l_n]$ with periodic boundary conditions (circle in $1-D$, torus in $2-D$, etc.). The regularity results also apply to homogeneous Neumann boundary conditions, since problems with Neumann boundary conditions can be extended to problems with periodic boundary conditions on a larger domain by gluing together copies of mirror images of the original rectangular. Regularity properties as in Lemma 2.1 are also known for Dirichlet or other boundary conditions (see e.g. Reference [18]). But additional boundary estimates are required and we are not aware of a general unifying formulation like Lemma 2.1 for that case.

In our proofs we use property (5) in about 12 different ways. To be as transparent as possible we enumerate the corresponding constants by C_j and the corresponding indices by $\alpha_j,\ j = 1, \ldots, 12$.

2.1. Local existence

To make the paper self contained we first show local existence. The local existence proof is generalizable to higher dimensions, hence we formulate it for arbitrary dimension $n \geqslant 1$. We study

$$\begin{aligned} u_t &= \Delta u - \nabla(\chi u \nabla v) \\ v_t &= \varepsilon \Delta v + u - av \end{aligned} \tag{6}$$

on a bounded smooth n-dimensional manifold M without boundary (which includes an interval $[0, l]$ with homogeneous Neumann or periodic boundary conditions).

Theorem 2.2
Consider $1 < \sigma < 2$ and $p > n/(\sigma - 1)$. We assume $u_0 \in L^\infty(\Omega)$ and $v_0 \in W^{\sigma,p}(\Omega)$. Then there exists a time $T > 0$ and a unique solution $(u(t), v(t))$ of (6) with

$$u \in L^\infty(0, T; L^\infty(\Omega)) \quad \text{and} \quad v \in L^\infty(0, T; W^{\sigma,p}(\Omega))$$

Although this result would follow from Amann's work [16] we give a proof in Section 3.

2.2. Global existence

For global existence we need some more indices and conditions for relations between these indices. To keep track of all these conditions we introduce a special notation:

Definition 2.3
A tupel of real parameters (σ, p, r, P, Q) is called *admissible*, if

$$1 < \sigma < 2, \quad \frac{1}{\sigma - 1} < p < \infty, \quad \frac{2p}{\sigma p + 1} < r < \frac{1}{\sigma - 1} \tag{7}$$

$$1 < P < 1 + \frac{1}{p}, \quad \frac{1}{P} + \frac{1}{Q} = 1, \quad \frac{1}{p} < \frac{Q}{r} < \frac{1}{p} + 2 \tag{8}$$

Indeed, the set of admissible parameters is not empty, since

$$(\sigma, p, r, P, Q) = (1.6, 2, 1.5, 1.4, 3.5)$$

is admissible, as is easily checked. Other admissible tupel can be found in the Appendix, where we also describe an algorithm to find admissible parameters.

Theorem 2.4
Let $n = 1$, $\Omega = [0, l]$ and suppose $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$ and $v_0 \in W^{\sigma,p}(\Omega)$, where σ, p belong to an admissible set of parameters. Then for each $T > 0$ there is a constant K which depends on $a, \varepsilon, \|u_0\|_1, \|u_0\|_\infty, \|v_0\|_{\sigma,p}, T$, and the admissible parameters (σ, p, r, P, Q) such that the solution of (1) with (2) (or (1) with (3)) satisfies

$$\sup_{0 \leqslant t \leqslant T} \|u(t)\|_\infty \leqslant K(T) \quad \text{and} \quad \sup_{0 \leqslant t \leqslant T} \|v(t)\|_{\sigma,p} \leqslant K(T)$$

The proof is given in Section 4.

Although K depends on T it is bounded for each finite T. This is shown at the end of the proof of Lemma 4.1.

3. PROOF OF LOCAL EXISTENCE

In this section we prove Theorem 2.2. We use Banach's fixed-point theorem on

$$\mathscr{X} = L^\infty(0, T; L^\infty(\Omega)) \times L^\infty(0, T; W^{\sigma,p}(\Omega))$$

Step 1. We consider a given function $z \in L^\infty(\Omega_T)$, where, as usual, $\Omega_T = (0, T) \times \Omega$ and we solve

$$v_t = \varepsilon \Delta v + z - av, \quad v(0) = v_0 \tag{9}$$

We set $w = e^{at} v$ and obtain a problem for w:

$$w_t = \varepsilon \Delta w + e^{at} z \tag{10}$$

Let $T_\varepsilon(t)$ denote the solution semigroup on M of

$$w_t = \varepsilon \Delta w \tag{11}$$

With a scaling of time $\tau = \varepsilon t$, Equation (11) is equivalent to

$$w_\tau = \Delta w \tag{12}$$

Hence with Lemma 2.1 we find that for $0 < \tau \leqslant 1$

$$T_1(\tau) : L^p \to W^{\sigma,p} \quad \text{with norm } C_1 \tau^{-\sigma/2}$$

which means that for $0 < t \leqslant 1/\varepsilon$

$$T_\varepsilon(t) : L^p \to W^{\sigma,p} \quad \text{with norm } C_1 \varepsilon^{-\sigma/2} t^{-\sigma/2} \tag{13}$$

We write the solution of (10) as

$$w(t) = T_\varepsilon(t)w(0) + \int_0^t T_\varepsilon(t-s)e^{as}z(s)\mathrm{d}s$$

and we use the above property (13) to estimate as follows:

$$\|w(t)\|_{\sigma,p} \leqslant \|w(0)\|_{\sigma,p} + C_1\varepsilon^{-\sigma/2}t^{1-\sigma/2}e^{at}\sup_t \|z(t)\|_\infty$$

This gives for $v = e^{-at}w$

$$\|v(t)\|_{\sigma,p} \leqslant e^{-at}\|v(0)\|_{\sigma,p} + C_1\varepsilon^{-\sigma/2}t^{1-\sigma/2}\sup_t \|z(t)\|_\infty \tag{14}$$

Step 2. With v from (14) we study the equation for u:

$$u_t = \Delta u - \chi\nabla u\nabla v - \chi u\Delta v, \quad u(0) = u_0 \tag{15}$$

which has the formal solution

$$u(t) = T_1(t)u_0 - \int_0^t T_1(t-s)\chi\nabla u\nabla v\,\mathrm{d}s - \int_0^t T_1(t-s)\chi u\Delta v\,\mathrm{d}s \tag{16}$$

We study the integral terms in (16) separately. The first integral term involves ∇v which is in $W^{\sigma-1,p}$. We use the Sobolev embedding

$$W^{\sigma-1,p} \hookrightarrow C^0, \quad \text{for } p > \frac{n}{\sigma-1}$$

We aim to use (16) to find an estimate for u in L^∞. If $u \in L^\infty$ then it is in L^q for $1 \leqslant q \leqslant \infty$. Then $\nabla u \in W^{-1,q}$ and we use Lemma 2.1 in the form

$$T_1(t): W^{-1,q} \to W^{1/2,q} \quad \text{with norm } C_2t^{-3/4} \tag{17}$$

where we choose q large enough so as to ensure $W^{1/2,q} \hookrightarrow C^0$.

Using (17) and the Sobolev embedding we find

$$\begin{aligned}\left\|\int_0^t T_1(t-s)\chi\nabla u\nabla v\,\mathrm{d}s\right\|_\infty &\leqslant \chi\sup_t \|\nabla v\|_\infty \left\|\int_0^t T_1(t-s)\nabla u\,\mathrm{d}s\right\|_\infty \\ &\leqslant \chi C_2t^{1/4}\sup_t \|v\|_{1,\infty}\sup_t \|u\|_\infty \\ &\leqslant \chi Ct^{1/4}\sup_t \|v\|_{\sigma,p}\sup_t \|u\|_\infty \end{aligned} \tag{18}$$

In the last estimate we used the fact that $W^{\sigma,p} \hookrightarrow C^1$ for $p > n/(\sigma-1)$.

In the second integral in (16) we encounter the term $\Delta v \in W^{\sigma-2,p}$. We use Lemma 2.1

$$T_1(t): W^{\sigma-2,p} \to W^{\sigma-1,p} \hookrightarrow C^0, \quad \text{with norm } C_3t^{-1/2} \tag{19}$$

Then we get

$$\left\|\int_0^t T_1(t-s)\chi u\Delta v\,\mathrm{d}s\right\|_\infty \leqslant \chi C_3t^{1/2}\sup_t \|v\|_{\sigma,p}\sup_t \|u\|_\infty \tag{20}$$

Putting (18) and (20) together we find from (16) that

$$\|u(t)\|_\infty \leqslant \|u_0\|_\infty + \chi C(t^{1/4}+t^{1/2})\sup_t \|v\|_{\sigma,p}\sup_t \|u\|_\infty \tag{21}$$

Together with the estimate for v, (16), we get

$$\begin{aligned}\|u(t)\|_\infty \leqslant {} & \|u_0\|_\infty + \chi C(t^{1/4}+t^{1/2})\sup_t \|u\|_\infty \\ & \cdot \left(e^{-at}\|v_0\|_{\sigma,p} + C_1\varepsilon^{-\sigma/2}t^{1-\sigma/2}\sup_t \|z(t)\|_\infty\right)\end{aligned} \tag{22}$$

We solve this for $\|u\|_{L^\infty(\Omega_T)}$ for some $T>0$ to find

$$\|u\|_{L^\infty(\Omega_T)} \leqslant \frac{\|u_0\|_\infty}{1-\chi C(T^{1/4}+T^{1/2})(e^{-aT}\|v_0\|_{\sigma,p} + C_1\varepsilon^{-\sigma/2}T^{1-\sigma/2}\|z(t)\|_{L^\infty(\Omega_T)})} \tag{23}$$

Step 3. We define a map $H : L^\infty(\Omega_T) \to L^\infty(\Omega_T)$ by $Hz = u$. Let $m > \|u\|_\infty$, then for T small enough

$$H : \mathscr{B}_m(0) \to \mathscr{B}_m(0)$$

where

$$\mathscr{B}_m(0) := \{\varphi \in L^\infty(\Omega_T) : \|\varphi\|_{L^\infty(\Omega_T)} < m, \quad \varphi(0) = u_0\}$$

Step 4. We show that, when T is small enough, the mapping H is a contraction on $\mathscr{B}_m(0)$. We consider two functions $z, Z \in \mathscr{B}_m(0)$ and we denote the images by $u = Hz, U = HZ$, respectively. The corresponding solutions of the v-equation (9) are denoted by v and V.

Since the v equation (9) is linear we can directly apply the estimate (14) and we obtain

$$\|v(t)-V(t)\|_{\sigma,p} \leqslant C_1\varepsilon^{-\sigma/2}t^{1-\sigma/2}\sup_t \|z(t)-Z(t)\|_\infty \tag{24}$$

The functions u, U are solutions of the u-equation (15) with v, V, respectively. We find for the difference

$$\begin{aligned}u(t)-U(t) = {} & -\int_0^t T_1(t-s)((\nabla u - \nabla U)\nabla v + \nabla U(\nabla v - \nabla V))\,\mathrm{d}s \\ & -\int_0^t T_1(t-s)((u-U)\Delta v + U(\Delta v - \Delta V))\,\mathrm{d}s\end{aligned}$$

For the integral terms we use the same estimates as before in (18) and (20). Then we obtain

$$\begin{aligned}\|u(t)-U(t)\|_\infty \leqslant {} & \chi(t^{1/4}+t^{1/2}) \\ & \cdot \left(\sup_t \|u-U\|_\infty \sup_t \|v\|_{\sigma,p} + \sup_t \|U\|_\infty \sup_t \|v-V\|_{\sigma,p}\right)\end{aligned} \tag{25}$$

From (14) we find a constant $\hat{C}$ such that

$$\sup_t \|v\|_{\sigma,p} \leqslant \hat{C}$$

Then it follows from (25) that

$$\sup_t \|u(t) - U(t)\|_\infty \leqslant \frac{\chi(T^{1/2} + T^{1/2})}{1 - \chi(T^{1/4} + T^{1/2})\hat{C}} \sup_t \|u\|_\infty \sup_t \|v - V\|_{\sigma,p}$$

Together with (24) we find that H is a contraction on $\mathscr{B}_m(0)$ when T is small enough.

Step 5. We apply Banach's fixed point theorem and obtain a unique fixed point of H. This corresponds to a unique weak solution $(u,v) \in \mathscr{X}$. □

4. PROOF OF GLOBAL EXISTENCE

In this section we prove Theorem 2.4. The first equation of (1) is a conservation equation which also preserves positivity. Hence if we denote

$$M := \int_0^l u_0(x)\,\mathrm{d}x$$

then we conclude that

$$\int_0^l u(t,x)\,\mathrm{d}x = M$$

as long as the solution exists.

Let (σ, p, r, P, Q) be an admissible set of parameters. First we use the L^1-bound to find a global Sobolev estimate for v:

Lemma 4.1
For each $0 < t \leqslant 1/\varepsilon$ we find

$$\|v(t)\|_{\sigma,r} \leqslant e^{-at}\|v_0\|_{\sigma,r} + C_4\varepsilon^{-\alpha_4}t^{1-\alpha_4}M \tag{26}$$

The parameters $C_4 > 0, 0 < \alpha_4 < 1$ are defined below. Moreover, for each $T > 0$ there is a constant $K_1(T)$ which depends on $a, \|v_0\|_{\sigma,r}, C_4, \varepsilon^{-\alpha_4}, M$ such that

$$\sup_{0 \leqslant t \leqslant T} \|v(t)\|_{\sigma,r} \leqslant K_1(T)$$

Proof
The formal solution of (10) where z is replaced by u is

$$w(t) = T_\varepsilon(t)w_0 + \int_0^t T_\varepsilon(t-s)e^{as}u(s)\,\mathrm{d}s \tag{27}$$

We like to have $T_\varepsilon u \in W^{\sigma,r}$. Using Lemma 2.1 we find that for $0 < t \leqslant 1/\varepsilon$

$$T_\varepsilon(t) : L^1 \to W^{\sigma,r} \quad \text{with norm } C_4(\varepsilon t)^{-\alpha_4} \tag{28}$$

with

$$-\alpha_4 = -\frac{\sigma}{2} + \frac{1}{2r} - \frac{1}{2}$$

We find that $0 > -\alpha_4 > -1$, since r belongs to an admissible tupel and it satisfies $r < 1/(\sigma - 1)$. We cannot choose $r = p$ here since we would then have $-\alpha_4 < -1$ and this would lead to a singularity in the following estimate. In case of $n = 2$ we would have to choose $r < 2/\sigma$, which would lead to contradictions in the definition of admissible. With the correct choice of r we find using (28)

$$\begin{aligned}\|w(t)\|_{\sigma,r} &\leqslant \|w_0\|_{\sigma,r} + C_4 t^{1-\alpha_4} \sup_t \|u(t)\|_{L^1} e^{at} \\ &= \|w_0\|_{\sigma,r} + C_4 t^{1-\alpha_4} \varepsilon^{-\alpha_4} e^{at} M\end{aligned}$$

With $v = e^{-at} w$ the first estimate in Lemma 4.1 is proven. To obtain the global *a priori* bound we repeat this estimate many times. If $T > 0$ is given then we choose some fixed increment $\theta \leqslant T/\varepsilon$ and apply the above estimate, (26), $N := [T/\theta] + 1$ times, where we choose the last iterate as the initial condition for the next iterate. The time increment is fixed and the constant $K_1(T)$ grows at most as $K_1(\theta)^N$. □

Now we look at the solution of the u equation of (1) and prove a global L^p estimate.

Lemma 4.2
For each $T > 0$ there is a constant $K_2(T)$, which depends on the admissible parameters (σ, p, r, P, Q), on $a, \|u_0\|, M$, and $K_1(T)$ such that

$$\sup_{0 \leqslant t \leqslant T} \|u(t)\|_p \leqslant K_2(T)$$

Proof
We use the formal solution for the u equation as given in (16). We aim to estimate $u(t)$ in L^p. For the first integral in (16) we use Young's inequality in the form

$$ab \leqslant \frac{1}{P} a^P + \frac{1}{Q} b^Q, \quad \text{with } \frac{1}{P} + \frac{1}{Q} = 1$$

where P, Q are part of an admissible tupel, as defined above. We find

$$\left\| \int_0^t T_1(t-s) \chi u_x v_x \, \mathrm{d}s \right\|_p \leqslant \left\| \int_0^t T_1(t-s) \frac{\chi}{P} u_x^P \, \mathrm{d}s \right\|_p + \left\| \int_0^t T_1(t-s) \frac{\chi}{Q} v_x^Q \, \mathrm{d}s \right\|_p \tag{29}$$

Since $u \in L^1$ we have $u_x^P \in W^{-1,1/P}$. We use Lemma 2.1 in the form that for each $0 < t \leqslant 1$

$$T_1(t) : W^{-1,1/P} \to L^p \quad \text{with norm } C_5 t^{-\alpha_5} \tag{30}$$

with

$$-\alpha_5 = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{p} - P\right)$$

We find that $0 > -\alpha_5 > -1$ since we assumed $1 + 1/p > P$ for admissible parameters. This estimate is not valid in higher dimensions $n \geqslant 2$. There we always find that $-\alpha_5 < -1$ which would, as mentioned earlier, lead to a singularity in the estimates. Here lies a significant difference to the higher dimensional case. With use of (30) we find that for $0 < t \leqslant 1$

$$\left\| \int_0^t T_1(t-s)\frac{\chi}{P}u_x^P \,\mathrm{d}s \right\|_p \leqslant C_5 t^{1-\alpha_5}\frac{\chi}{P}M^P \leqslant C_5\frac{\chi}{P}M^P \tag{31}$$

For the second term in (29) we use the fact that $v \in W^{\sigma,r}$ which implies that $v_x^Q \in W^{\sigma-1,r/Q} \subset L^{r/Q}$. We use Lemma 2.1 in the form that for each $0 < t \leqslant 1$

$$T_1(t) : L^{r/Q} \to L^p \quad \text{with norm} \quad C_6 t^{-\alpha_6} \tag{32}$$

with

$$-\alpha_6 = \frac{1}{2}\left(\frac{1}{p} - \frac{Q}{r}\right)$$

We find that $0 > -\alpha_6 > -1$ since $1/p < Q/r < 2 + 1/p$, as required for admissible parameters. Hence we get that for $0 < t \leqslant 1$

$$\left\| \int_0^t T_1(t-s)\frac{\chi}{Q}v_x^Q \,\mathrm{d}s \right\|_p \leqslant C_6 t^{1-\alpha_6} \sup_t \|v\|_{\sigma,r}^Q \leqslant C_6 K_1^Q(1) \tag{33}$$

where we used the *a priori* estimate of the previous Lemma 4.1.

We also use Young's inequality for the second integral term of (16).

$$\left\| \int_0^t T_1(t-s)\chi u v_{xx} \,\mathrm{d}s \right\|_p \leqslant \left\| \int_0^t T_1(t-s)\frac{\chi u^2}{2} \,\mathrm{d}s \right\|_p + \left\| \int_0^t T_1(t-s)\frac{\chi v_{xx}^2}{2} \,\mathrm{d}s \right\|_p \tag{34}$$

Since $u \in L^1$ we have $u^2 \in L^{1/2}$ and we use Lemma 2.1 in the form that for each $0 < t \leqslant 1$

$$T_1(t) : L^{1/2} \to L^p \quad \text{with norm} \quad C_7 t^{-\alpha_7} \tag{35}$$

with

$$-\alpha_7 = -1 + \frac{1}{2p}$$

Here we see that the choice of $p = \infty$ will not work directly, because then $-\alpha_7 = -1$. For admissible $p < \infty$ we find that for $0 < t \leqslant 1$

$$\left\| \int_0^t T_1(t-s)\frac{\chi u^2}{2} \,\mathrm{d}s \right\|_p \leqslant C_7 \chi t^{1/(2p)} M^2 \leqslant C_7 M^2 \tag{36}$$

For the second integral in (34) we observe $v_{xx}^2 \in W^{\sigma-2,r/2}$, hence we use Lemma 2.1 in the form that for each $0 < t \leqslant 1$

$$T_1(t) : W^{\sigma-2,r/2} \to L^p \quad \text{with norm } C_8 t^{-\alpha_8} \tag{37}$$

with

$$-\alpha_8 = \frac{\sigma - 2}{2} + \frac{1}{2}\left(\frac{1}{p} - \frac{2}{r}\right)$$

Since the parameters are admissible we have $-1 < -\alpha_8 < 0$. Then for $0 < t \leqslant 1$ we get

$$\begin{aligned}\left\| \int_0^t T_1(t-s)\frac{\chi v_{xx}^2}{2}\,\mathrm{d}s \right\|_p &\leqslant C_8 \chi t^{1-\alpha_8} \sup_t \|v\|_{\sigma,r}^2 \\ &\leqslant C_8 \chi K_1(1) \end{aligned} \tag{38}$$

We consider the general solution (16) and collect all of the above estimates (29), (31), (33), (34), (36), and (38) and find that for all $0 < t \leqslant 1$:

$$\|u(t)\|_p \leqslant \|u_0\|_p + \chi\left(C_5 \frac{M^P}{P} + C_6 K_1^Q(1) + C_7 M^2 + C_8 K_1(1)^2\right) \tag{39}$$

The right-hand side of this estimate only depends on powers of M and on powers of $K_1(1)$. We can iterate this estimate (as we did before in the proof of Lemma 4.1) and find the global estimate as stated in Lemma 4.2. □

In Lemma 4.2 we were able to generate a global L^p estimate for u from the particle conservation property. Notice that the case of $p = 2$ is included, since $p = 2$ belongs to an admissible set of parameters. Hence the estimate of Childress and Percus [9] is included here. We use the L^p estimate to find a global $W^{\sigma,p}$ bound for v:

Lemma 4.3
For each $T > 0$ there is a constant $K_3(T)$, which depends on the admissible parameters (σ, p, r, P, Q), on $a, \|v_0\|_{\sigma,p}, M$, and $K_2(T)$ such that

$$\sup_{0 \leqslant t \leqslant T} \|v(t)\|_{\sigma,p} \leqslant K_3(T) \tag{40}$$

Proof
Again we use Equation (27) for w and we use Lemma 2.1, that for $0 < t \leqslant 1$

$$T_\varepsilon(t) : L^p \to W^{\sigma,p} \quad \text{with norm } C_9(\varepsilon t)^{-\sigma/2} \tag{41}$$

Then with use of the previous Lemma 4.2 we find for $0 < t \leqslant 1$ that

$$\sup_{0 \leqslant s \leqslant t} \|w(s)\|_{\sigma,p} \leqslant \|w_0\|_{\sigma,p} + C_9 t^{1-\sigma/2} \varepsilon^{-\sigma/2} K_2(t) \tag{42}$$

Again we iterate this estimate (as done above) and obtain (40). □

Now we reach the level of regularity as required by the local existence result. We finally need to estimate $\|u\|_\infty$.

Lemma 4.4
For each $T > 0$ there is a constant $K_4(T)$, which depends on the admissible parameters (σ, p, r, P, Q), on $a, \|u_0\|_\infty, M$, and $K_2(T), K_3(T)$ such that

$$\sup_{0 \leqslant t \leqslant T} \|u(t)\|_\infty \leqslant K_4(T)$$

Proof
We use the formal solution as given in (16) and estimate

$$\|u(t)\|_\infty \leqslant \|u_0\|_\infty + \left\| \int_0^t T_1(t-s)\chi u_x v_x \, \mathrm{d}s \right\|_\infty + \left\| \int_0^t T_1(t-s)\chi u v_{xx} \, \mathrm{d}s \right\|_\infty \tag{43}$$

For the first integral term on the right-hand side we use the fact that $v_x \in W^{\sigma-1,p} \hookrightarrow C^0$. And we use Lemma 2.1. For $0 < t \leqslant 1$ we have

$$T_1(t) : W^{-1,p} \to W^{\sigma-1,p} \quad \text{with norm } C_{10} t^{-\sigma/2} \tag{44}$$

Then we get for $0 < t \leqslant 1$ that

$$\left\| \int_0^t T_1(t-s)\chi u_x v_x \, \mathrm{d}s \right\|_\infty \leqslant C_{10} \chi t^{1-\sigma/2} \sup_t \|u\|_p \sup_t \|v\|_{\sigma,p} \tag{45}$$

For the second integral term on the right-hand side of (43) we use again Young's inequality to obtain

$$\left\| \int_0^t T_1(t-s)\chi u v_{xx} \, \mathrm{d}s \right\|_\infty \leqslant \left\| \int_0^t T_1(t-s)\frac{\chi u^2}{2} \, \mathrm{d}s \right\|_\infty + \left\| \int_0^t T_1(t-s)\frac{\chi v_{xx}^2}{2} \, \mathrm{d}s \right\|_\infty \tag{46}$$

For the first integral term in (46) we now use the fact that $u^2 \in L^{p/2}$ and that

$$T_1(t) : L^{p/2} \to W^{\sigma-1,p} \hookrightarrow C^0 \quad \text{with norm } C_{11} t^{-\alpha_{11}} \tag{47}$$

with

$$-\alpha_{11} = -\frac{\sigma-1}{2} - \frac{1}{2p}$$

Note that $0 > -\alpha_{11} > -1$, since $\sigma > 1, p > 1$ as required for admissible parameters. We obtain for $0 < t \leqslant 1$ that

$$\left\| \int_0^t T_1(t-s)\frac{\chi u^2}{2} \, \mathrm{d}s \right\|_\infty \leqslant C_{11} \chi t^{1-\alpha_{11}} \sup_t \|u\|_p^2 \tag{48}$$

Finally, for the last term in (46) we find that for $0 < t \leqslant 1$

$$T_1(t) : W^{\sigma-2,p/2} \to W^{\sigma-1,p} \hookrightarrow C^0 \quad \text{with norm } C_{12} t^{-\alpha_{12}} \tag{49}$$

with

$$-\alpha_{12} = -\frac{1}{2} - \frac{1}{2p}$$

which is larger than -1 for $p>1$. We obtain for $0<t\leqslant 1$

$$\left\| \int_0^t T_1(t-s)\frac{\chi v_{xx}^2}{2}\,\mathrm{d}s \right\|_\infty \leqslant C_{12}\chi t^{1/2-1/(2p)} \sup_t \|v\|_{\sigma,p}^2 \tag{50}$$

To complete the proof of Lemma 4.4 we combine all estimates (45), (46), (48), (50) with the estimate for u, (43), and we use the previous Lemmas 4.2, and 4.3 to obtain the global bound for u in L^∞. □

Theorem 2.4 is a direct application of the *a priori* estimates of Lemmas 4.3 and 4.4 to the solution found in Theorem 2.2.

5. NUMERICAL RESULTS AND ASYMPTOTIC PROFILE

Numerical solutions of system (1) were considered by many authors (see e.g. References [9,19]). For small values of $\varepsilon>0$ numerical methods suggest that the solution might blow up in finite time. In the previous section we have proven that the solution stays bounded and exists globally for all $\varepsilon>0$. In this section we use scaling arguments and numerical simulations to investigate the growth characteristics of the solutions.

Numerical experiments show that the time evolution of this system can be split into three stages: (i) a *transient stage*, with slow growth of local maxima, (ii) a *blow-up stage*, where the solution grows very fast to a single peak distribution and (iii) a *saturation stage*, where the solution converges to a peak solution. We use perturbation analysis and numerical methods to find the asymptotical profile for

$$\varepsilon u\left(\frac{x}{\varepsilon}\right)$$

Let us first examine some simulations. We choose the domain length $l=1$. Then the system (1) has a constant solution $u_0=M$, $v_0=M/a$, and it is linearly unstable to perturbations of the form $A\cos(\pi kx)$ provided

$$0<k^2<\frac{M\chi-a}{\varepsilon\pi^2}$$

We choose $M=4$, $\chi=1$, $a=1$, then the homogeneous solution loses stability for $\varepsilon\leqslant 0.3$. Here we use various values for $\varepsilon\leqslant 0.1$. In Figure 1 we show a typical time evolution from two-, (a), or one-peak, (b), initial data with small amplitude.

Several modes are unstable and a pattern with several peaks starts to appear. Such patterns are unstable and eventually only a single peak remains, typically at the boundary. For this reason we study the growth of such a peak in more detail and we choose the initial condition

$$u=M+0.1\cos(\pi x)$$

which has one maximum at $x=0$. From a sequence of calculations for various ε we find that the final peak height is approximately proportional to $1/\varepsilon$, and the final width is proportional to ε. This motivates a scaling of the form $\varepsilon u(x/\varepsilon)$. In Figure 2 we show these rescaled profiles for various values of ε. As ε decreases, the rescaled profiles approximate a limiting curve

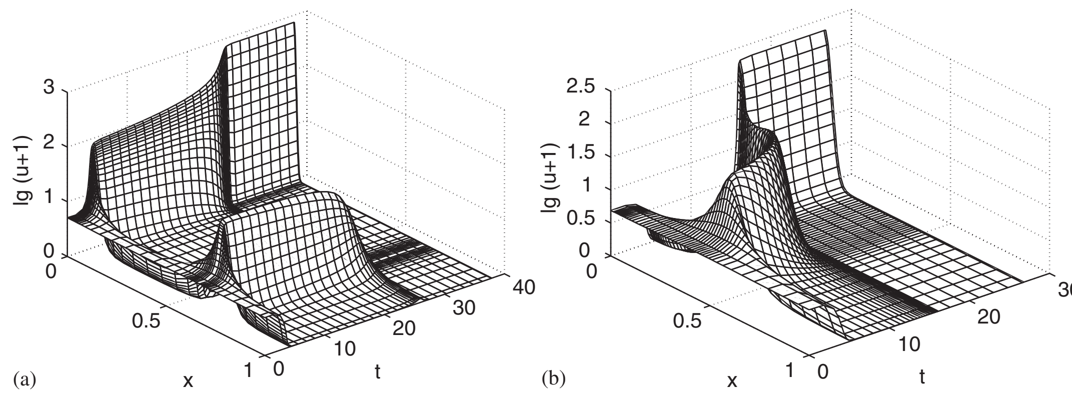

Figure 1. Typical time evolution for initial data with two peaks (a) and one peak (b) with small amplitude relative to a homogeneous solution. Note that u is shown on a logarithmic scale.

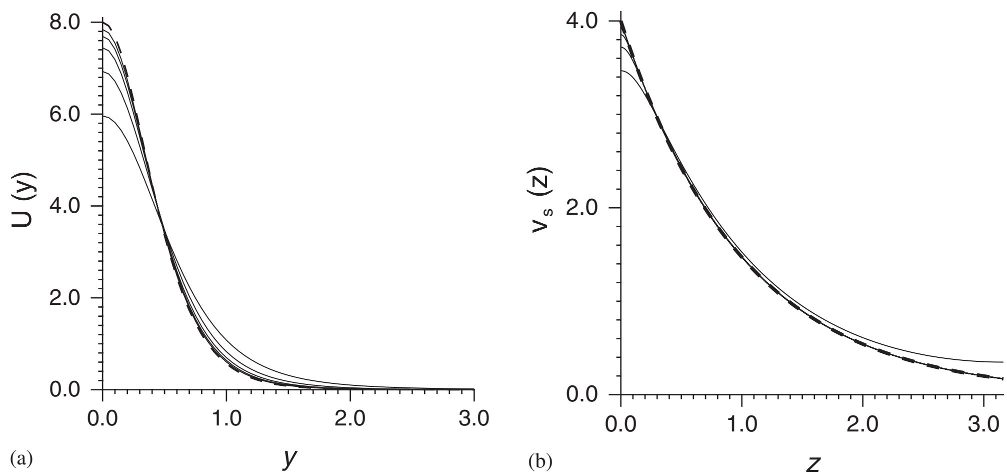

Figure 2. (a) The rescaled profiles of $U(y) = \varepsilon u(x/\varepsilon)$ for $\varepsilon = 0.1, 0.033, 0.01, 0.0033, 0.001$, the dashed line corresponds to the asymptotic stationary profile $U_0(y)$ (59); and (b) the profiles of $\sqrt{a\varepsilon}V(z)$ for the same values of ε compared to the asymptotic profile $v_s(z)$ as given in (62) (dashed line), where $z = \sqrt{\frac{a}{\varepsilon}}x$.

$U_0(y)$ which is given explicitely in (59). Moreover, we also find an approximate profile for $V(y)$ in (62).

To investigate the growth of these peaks over time we show in Figure 3 the time evolution of the maximum of the solution for different values of ε. For small ε we observe an initial transient phase, a steep growth phase and a saturation phase. Solutions grow to the height of ε^{-1} with a growth rate of $\varepsilon^{-2}/\ln(1/\varepsilon)$. In Figure 4 we show a plot of the time derivative,

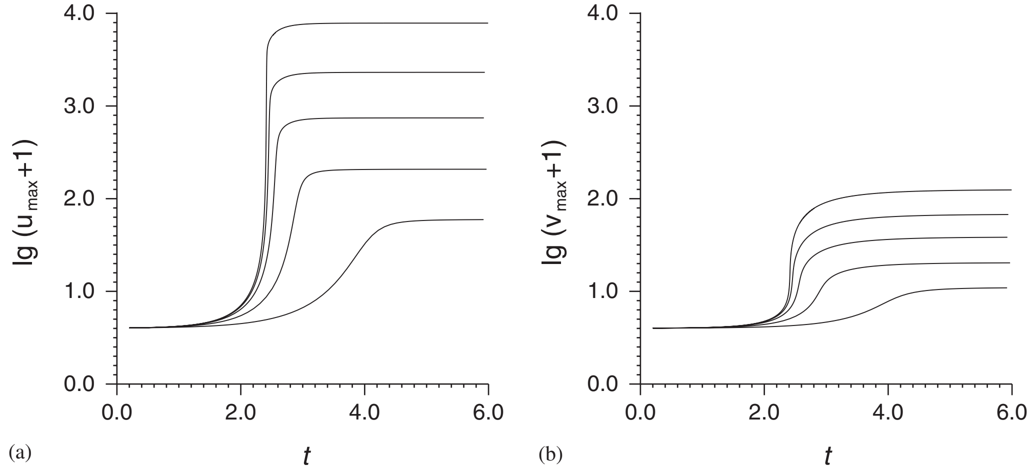

Figure 3. (a) Height of the peak u as a function of time for $\varepsilon = 0.1, 0.033, 0.01, 0.0033, 0.001$, (bottom to top on graph, respectively). The following features can be seen: (1) there are three stages of growth—slow, explosive (note the logarithmic scale), and then slowing and convergence to a stationary state; (2) the height of the peak scales as $\sim 1/\varepsilon$; and (b) Trajectories of the maximum of v for the same ε values as in a).

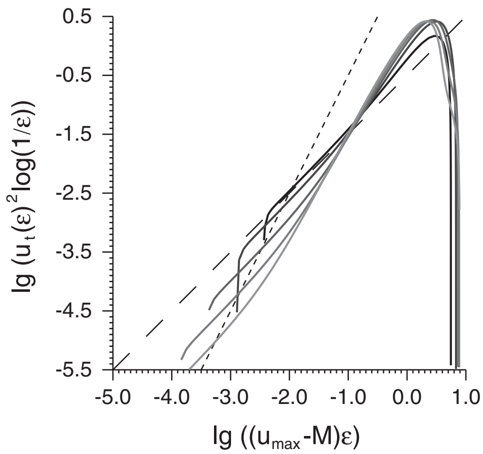

Figure 4. Trajectories of the peak growth for $\varepsilon = 0.1, 0.033, 0.01, 0.0033, 0.001$, lighter shadow of grey corresponds to smaller ε. The long dashed line shows exponential growth $u_t \sim u$, short-dashed line shows blow up growth $u_t \sim u^2$. After short transient period u first grows exponentially, then it accelerates and corresponds to $u_t \sim u^2$. After reaching the height ε^{-1} the growth slows down and stops. The smaller ε, the greater is the largest growth rate.

$\ln(u_t \varepsilon^2 \ln(1/\varepsilon))$ as a function of $\ln((u_{\max} - M)/\varepsilon)$. The short-dashed line corresponds to exponential growth $u_t \sim u$ and the long-dashed line corresponds to quadratic growth $u_t \sim u^2$. We clearly observe quadratic growth during the time evolution.

5.1. *Asymptotic profile for u*

To obtain the shape of the asymptotic profile we change coordinates to local space variables, $y = x/\varepsilon$ and fast time scale $\tau = t/\varepsilon$. We introduce new functions

$$u = \frac{U(y,\tau)}{\varepsilon}, \quad v = V(y,\tau)$$

Then $\partial/\partial t = \varepsilon^{-1}\partial/\partial\tau$, and $\partial/\partial x = \varepsilon^{-1}\partial/\partial y$, and Equations (1) take the form

$$\varepsilon U_\tau = U_{yy} - \chi(UV_y)_y \tag{51}$$
$$V_\tau = V_{yy} + U - \varepsilon a V \tag{52}$$
$$0 < y < L = \varepsilon^{-1}$$

with the boundary conditions $U_y = V_y = 0$ for $x = 0$ and $x = L = \varepsilon^{-1}$. The first equation of (1) is a conservation law

$$\int_0^1 u(x,t)\mathrm{d}x = \int_0^L U(y,\tau)\mathrm{d}y = M \tag{53}$$

We assume that there is a universal stationary profile $\hat{U}(y)$, which describes the shape of the stationary peak. If we set $\hat{U}_\tau = V_\tau = 0$, then

$$\hat{U}_{yy} - \chi(\hat{U}V_y)_y = 0 \tag{54}$$
$$V_{yy} + \hat{U} - a\varepsilon V = 0 \tag{55}$$

We study the formal expansions

$$\hat{U}(y) = U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y), \quad V(y) = V_0(y) + \varepsilon V_1(y) + \varepsilon^2 V_2(y) \tag{56}$$

Substituting (56) into (54), (55) we get the leading order:

$$U_{0yy} - \chi\left(U_0 V_{0y}\right)_y = 0 \tag{57}$$
$$V_{0yy} + U_0 = 0 \tag{58}$$

From the first equation it follows that $U_{0y} - \chi U_0 V_{0y} = C_1$, where $C_1 = 0$ according to the boundary conditions. After second integration we obtain

$$U_0 = Ae^{\chi V_0}$$

where $A \geqslant 0$ is a constant. We assume that all the mass is contained in the leading-order term, hence

$$A\int_0^\infty e^{\chi V_0(y)}\mathrm{d}y = M$$

Substituting U_0 into equation (58) we get

$$V_{0yy} + Ae^{\chi V_0} = 0$$

Integrating once we find

$$\frac{(V_{0y})^2}{2} + Ae^{\chi V_0} = E$$

We integrate this separable equation directly and find the relation

$$\sqrt{\frac{E}{A}} e^{-\chi V_0/2} = \cosh\left(\chi\sqrt{\frac{E}{2}}(y - y_0)\right)$$

This gives

$$U_0 = Ae^{\chi V_0} = \frac{E}{(\cosh(q(y - y_0)))^2} \quad \text{with } q = \chi\sqrt{\frac{E}{2}}$$

Now let us assume that the centre of the peak is at $y = 0$, then $y_0 = 0$. Using (53) we have

$$M = \int_0^\infty \frac{E\,\mathrm{d}y}{(\cosh(qy))^2} = \frac{E}{q}\int_0^\infty \frac{\mathrm{d}z}{(\cosh z)^2} = \frac{1}{\chi}\sqrt{2E}$$

for large values of L. Therefore,

$$E = \frac{\chi^2 M^2}{2}, \quad q = \frac{\chi^2 M}{2}$$

and we come to the asymptotic peak shape

$$\hat{U}(y) \approx U_0(y) = \frac{qM}{(\cosh(qy))^2} \tag{59}$$

Figure 2 shows that the asymptotic form of the peak is described by $U_0(y)$ quite well.

In the perturbation analysis, as used here, we cannot obtain V_0 uniquely. The leading-order term V_0 is given by $V_0 = \chi^{-1}\ln U_0 - \ln A$, hence it depends upon the constant A. In our perturbation analysis (58) we lose the decay term for v. Hence there is no unique solution for V_0.

5.2. *Asymptotic profile for v*

To find an asymptotic profile for v we use the fact that the time evolution ends up at a stationary state. Hence we can find the final stationary profile of v_s from the asymptotic profile of u (59) by the equation

$$\varepsilon v_{sxx} - av_s = -\hat{U}(x/\varepsilon)/\varepsilon \tag{60}$$

Unfortunately, we cannot solve this equation. Nonetheless, it is possible to find an approximate solution for small ε.

The solution of $\varepsilon v_{xx} - av = 0$ behaves like e^{-kx}, $k = \sqrt{a/\varepsilon}$, that is we can expect that the peak of v has a width proportional to $\sim \sqrt{\varepsilon}$. On the other hand, the width of the u peak is $\sim \varepsilon$. Therefore, for small values of ε we can assume that, from the viewpoint of the v-component, all mass is already concentrated at $x=0$ and $u(x) \approx 2M\delta_0(x)$. The factor 2 appears because the domain $x \geqslant 0$ contains only half of the δ-peak, the whole δ-peak carries mass $2M$, where

$$M = \int_0^L u(x,t)\mathrm{d}x$$

Then (60) becomes

$$\varepsilon v_{sxx} - av_s = -2M\delta_0(x) \tag{61}$$

For $\varepsilon \to 0$ we can use the solution for the infinite domain

$$v_s(x) \cong \frac{M}{\sqrt{a\varepsilon}} \exp\left(-\sqrt{\frac{a}{\varepsilon}}|x|\right) \tag{62}$$

Note that this v_s does not satisfy condition $v_x(0)=0$. To investigate v close to 0 we would have to use Equation (60).

We compare this function $v_s(x)$ with the profiles of $v(x)$ from our numerical simulations in Figure 2(b). The dashed line is $\sqrt{a\varepsilon}v_s(z) = Me^{-|z|}$, $z=\sqrt{a/\varepsilon}x$, and there is a good agreement with numerically obtained rescaled profiles of $\sqrt{a\varepsilon}v(z)$ as $\varepsilon \to 0$.

The fact that approximation (62) works well gives a simple heuristic explanation of why the blow up stops. When the width of the u profile is of the order ε, the profile of v practically ceases to change. It will not grow further even if u becomes a δ-function. As v does not change, u adjusts itself to the v profile and does not grow any more either.

This argument does not work in higher dimensions. For $n \geqslant 2$ the above equation (61) can be formally solved using the corresponding Greens function. The solution shows a term of the form $\sim G(x,y) * \delta_0(y)$, which $\to +\infty$ for $\|x\| \to 0$. Hence in higher dimensions u and v will grow simultaneously. This leads to a blow up as already shown earlier.

APPENDIX A: ADMISSIBLE PARAMETERS

Other admissible parameters with integer p, r, and Q are $(1.3, 4, 3, 1.2, 6)$, or $(1.1, 16, 9, 1.058, 18)$, for example. We estimated numerically that the set of all admissible tupels is contained in

$$\{1<\sigma<2, 1<p<\infty, 2/3<r<\infty,\ 1<P<2, 2<Q<\infty\}.$$

We are grateful to Guangjun Cao [20] who figured out the following procedure to find admissible parameters:

Step 1: Choose σ with $1<\sigma<2$,

Step 2: choose p with $\frac{1}{\sigma-1} < p < \frac{1}{2(\sigma-1)}\left(3-\sigma+\sqrt{4+(\sigma-1)^2}\right)$,

Step 3: choose r with $\frac{p^2+p}{1+2p} < r < \frac{1}{\sigma-1}$,

Step 4: choose P with $\frac{r(1+2p)}{r(1+2p)-p} < P < 1+\frac{1}{p}$,

Step 5: define Q by $Q=\frac{P}{P-1}$.

It requires quite a number of elementary calculations to show that the above conditions are not self contradictory and to show that parameters found by this procedure are indeed admissible in the sense of Definition 2.3. We checked it carefully.

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