(3.5) First-order PDE's, Method of Characteristics

A linear first-order PDE

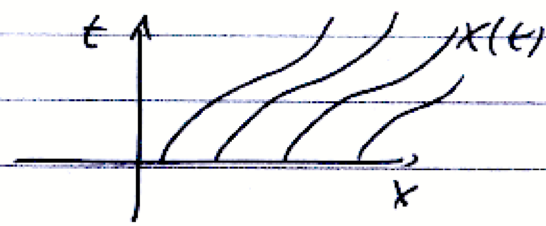
$$\frac{\partial z(x,t)}{\partial t} - c\,\frac{\partial z(x,t)}{\partial x} = 0 \qquad (1)$$

$$z(x,0) = f(x)$$

for $-\infty < x < +\infty$, $t \geq 0$

no boundary conditions needed, since we work on an unbounded domain.

Idea: Find curves $x(t)$ in the (x,t)-plane, such that the PDE can be reduced to an ODE on these curves.
$x(t)$: characteristics.



Assume we can write a solution like this:
$z = z(x(t), t)$.
The chain-rule gives $\frac{dz}{dt}(x(t),t) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial t}$

Compare this to equation (1).
If we choose $x(t)$ to satisfy $\frac{\partial x}{\partial t} = -c$,

then we get $\frac{dz}{dt}(x(t),t) = -c\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0$

Hence we have transformed one PDE (1) into two ODE's (<u>characteristic equations</u>)

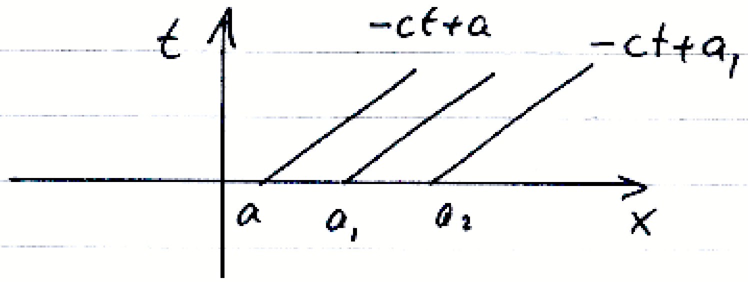
$$\frac{\partial x(t)}{\partial t} = -c \quad , \quad \frac{dz(x(t),t)}{dt} = 0$$

Now we solve the first one:

$$x(t) = -ct + a \qquad a \text{ constant.}$$

a is the initial point of the characteristic curve
$-c$ is it's slope:

$$a = x + ct$$



Now, solve the second characteristic equation:

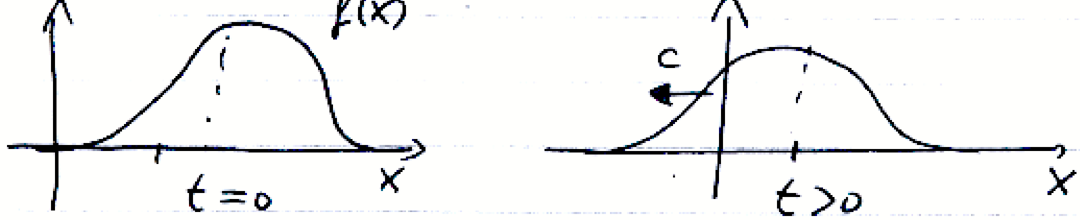
$$\frac{d}{dt} z(x(t),t) = 0 \quad \Rightarrow \quad z(x(t),t) = \text{const.}$$

In particular $z(x(t),t) = z(x(0),0) = f(x(0)) = f(a)$
$= f(x+ct)$

Now assume (x,t) is given, then

Solution: $\boxed{z(x,t) = f(x+ct).}$

Which is a transition to the left with velocity c:



Example 1: Consider
$$\left.\begin{aligned} &\frac{\partial z}{\partial t} + 5\frac{\partial z}{\partial x} = 0 \\ &z(x,0) = e^{-x^2} \end{aligned}\right\}$$

(i) Write down the characteristic equations and solve them

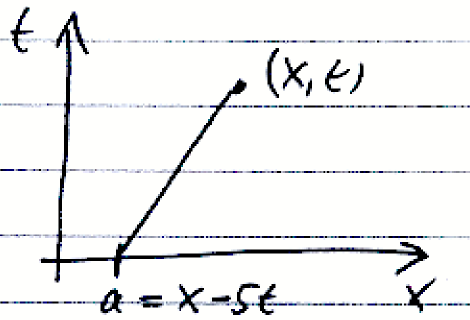
(ii) Given (x,t), find the solution $z(x,t)$.

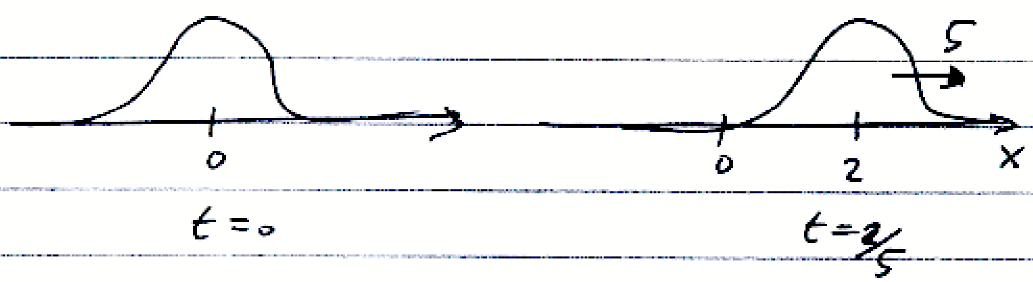
(i) $$\frac{dx(t)}{dt} = 5 \quad \Rightarrow \quad x(t) = 5t + a \,, \text{ or } \; a = x - 5t$$

$$\frac{dz(x(t),t)}{dt} = 0 \quad \Rightarrow \quad z(x(t),t) = f(a) = e^{-(x-5t)^2}$$

(ii) given (x,t) then

$$z(x,t) = e^{-(x-5t)^2}$$





Now with source (or sink) terms:

$$\boxed{\frac{\partial u(x,t)}{\partial t} + \alpha \frac{\partial u(x,t)}{\partial x} + \beta u(x,t) = 0} \qquad (2)$$

Again, we look for solutions $u(x(t),t)$.
If $u(x(t),t)$ is a solution, then from the chain-rule

$$\frac{d}{dt} u(x(t),t) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial t}$$

it follows that if $\frac{dx}{dt} = \alpha$, then $\frac{d}{dt} u(x(t),t) = -\beta u(x(t),t)$

Hence we get the two <u>characteristic equations</u>

$$\frac{dx(t)}{dt} = \alpha, \qquad \frac{d}{dt} u = -\beta u \quad , \; u(t) = u(x(t),t)$$

Then $x(t) = \alpha t + a$ $\qquad u(t) = u(0) e^{-\beta t}$

Given (x,t), then $a = x - \alpha t$ and

$$u(x,t) = u(x(0),0) e^{-\beta t} = \underline{\underline{f(x-\alpha t) e^{-\beta t}}}$$

<u>Example 2:</u> Solve $\left.\begin{array}{l} \frac{\partial u}{\partial t} + \sqrt{3}\frac{\partial u}{\partial x} - 16u = 0 \\ u(x,0) = \sin^2(x) \end{array}\right\}$

Just apply the above procedure:

characteristic equations

$\frac{dx(t)}{dt} = \sqrt{3}$, $x(t) = \sqrt{3}\,t + a$, $a = x - \sqrt{3}\,t$

$\frac{du(t)}{dt} = 16u(t)$, $u(t) = u(0)\,e^{16t}$

Solution $u(x,t) = \sin^2(x - \sqrt{3}\,t)\,e^{16t}$

Example 3 $\quad \dfrac{\partial w}{\partial t} - x\dfrac{\partial w}{\partial x} = 0 \qquad 0 < t < \infty, \quad -\infty < x < \infty$

$$w(x,0) = x^3 - 1$$

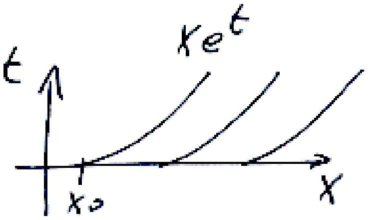
Method of characteristics: $\quad x(t): \quad w = w(x(t), t)$

$$\frac{d}{dt} w(x(t), t) = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial t}$$

and $\dfrac{\partial x}{\partial t} = -x(t) \implies x(t) = x_0 e^{-t} \quad$ or $\quad x_0 = x e^{t}$

Then

$$\frac{d}{dt} w(x(t), t) = 0$$



$$\begin{aligned} w(x(t), t) &= \text{constant} \\ &= w(x(0), 0) \\ &= x_0^3 - 1 \\ &= (xe^t)^3 - 1 = x^3e^{3t} - 1 \end{aligned}$$

solution: $\boxed{w(x,t) = x^3 e^{3t} - 1}$

test: $w(x,0) = x^3 - 1$ ✓

$$\frac{\partial w}{\partial t} = 3x^3e^{3t}, \qquad \frac{\partial w}{\partial x} = 3x^2e^{3t}$$

$$\frac{\partial w}{\partial t} - x\frac{\partial w}{\partial x} = 0 \quad ✓$$

Example 4
$$\left.\begin{aligned} \frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} &= \sin(2\pi t) \\ z(x,0) &= \cos x \end{aligned}\right\}$$

Solve this initial value problem on $-\infty < x < \infty$.

$z(x(t), t)$. $\quad \frac{\partial x}{\partial t} = 3 \Rightarrow x(t) = 3t + a$ or $a = x - 3t$.

Then
$$\frac{d z(x(t),t)}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial t} = \sin(2\pi t)$$

$$\Rightarrow \quad z(x(t), t) = -\frac{1}{2\pi}\cos(2\pi t) + C_1$$

$$z(x(0), 0) = -\frac{1}{2\pi} + C_1 = \cos(a)$$

$$= \cos(x - 3t)$$

$$\Rightarrow \quad C_1 = \cos(x-3t) + \frac{1}{2\pi}$$

solution: $\boxed{z(x,t) = -\frac{1}{2\pi}\cos(2\pi t) + \cos(x-3t) + \frac{1}{2\pi}}$

test: $z(x,0) = -\frac{1}{2\pi} + \cos x + \frac{1}{2\pi} = \cos x$

$$\frac{\partial z}{\partial t} = \sin(2\pi t) + 3\sin(x-3t)$$

$$\frac{\partial z}{\partial x} = -\sin(x-3t)$$

Hence $\frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} = \sin(2\pi t)$