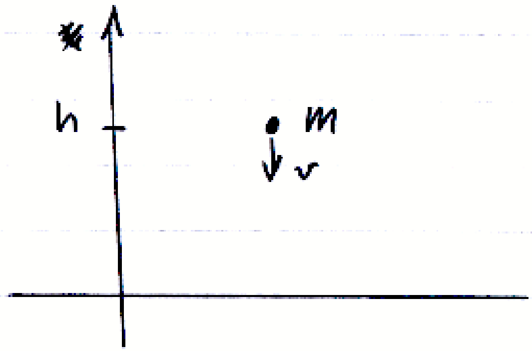
3.4) Conservation of energy



kinetic energy: $E_k = \frac{1}{2} m v^2$

potential energy: $E_p = \frac{1}{2} k h^2$

We study the following homogeneous Neumann problem for the wave equation:

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{on } [0,L] \\ &u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x) \\ &\frac{\partial u(0,t)}{\partial x} = 0 \quad \frac{\partial u(L,t)}{\partial x} = 0 \end{aligned}\right\}$$

Here the kinetic energy is $\quad E_k = \frac{1}{2} \int_0^L \left(\frac{\partial u(x,t)}{\partial t} \right)^2 dx$

and the potential energy is $\quad E_p = \frac{c^2}{2} \int_0^L \left(\frac{\partial u(x,t)}{\partial x} \right)^2 dx$

total energy $\quad E = E_k + E_p$

We want to show $\quad \frac{dE}{dt} = 0 \qquad$ (hence $E = \text{const.}$)

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2} \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \right]$$

$$= \frac{d}{dt}\left[\frac{1}{2} \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 + c^2 \left(\frac{\partial u}{\partial x}\right)^2 dx \right]$$

Leipniz

$$= \frac{1}{2} \int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 + c^2 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 \, dx$$

$$= \int_0^L \frac{\partial u}{\partial t} \cdot \frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} \, dx$$

now we use the wave equation

$$= \int_0^L c^2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} \, dx$$

now observe $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} \frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial t} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}$

$$= c^2 \int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} \frac{\partial u}{\partial x}\right) dx$$

$$= c^2 \left[\frac{\partial u}{\partial t} \frac{\partial u}{\partial x}\right]_0^L$$

$= 0$ from the homogeneous Neumann conditions.

<u>Corollary</u>: (Uniqueness for the ~~[illegible]~~ Neumann problem)

~~[illegible]~~ The Neumann problem for the wave equation has at most <u>one</u> solution.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{on } [0, L]$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$\frac{\partial u}{\partial x}(0,t) = g_1(t), \quad \frac{\partial u}{\partial x}(L,t) = g_2(t)$$

Proof (energy method) Assume two solutions $u_1(x,t)$, $u_2(x,t)$

define $w(x,t) = u_1(x,t) - u_2(x,t)$.
Then $w(x,t)$ satisfies the initial boundary value problem

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

$$w(x,0) = 0, \quad \frac{\partial w(x,0)}{\partial t} = 0$$

$$\frac{\partial w}{\partial x}(0,t) = 0, \quad \frac{\partial w}{\partial x}(L,t) = 0$$

From conservation of total energy we know that

$$E(t) = \text{const} = E(0).$$

$$\frac{1}{2}\int_0^L \left(\frac{\partial w(x,t)}{\partial t}\right)^2 + \frac{c^2}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx = \frac{1}{2}\int_0^L \Big(\underbrace{\frac{\partial w(x,0)}{\partial t}}_{=0}\Big)^2 + \frac{c^2}{2}\Big(\underbrace{\frac{\partial w(x,0)}{\partial x}}_{=0}\Big)$$

$$= 0$$

Hence $\frac{\partial w}{\partial t} \equiv 0$, $\frac{\partial w}{\partial x} \equiv 0$

Finally $w(x,t) = 0 \Rightarrow u_1 = u_2$ $\square$