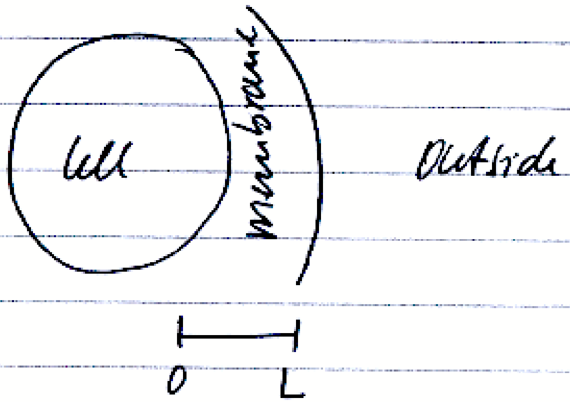
(2.5) Steady States and Equilibrium solutions

Definition A <u>steady-state</u> or <u>equilibrium</u> solution of an initial-boundary value problem of a PDE is a solution that does not depend on time, $u(t,x) = \bar{u}(x)$

Example 1 (Diffusion through a cell-membrane)



$u(t,x)$: concentration of oxygene.

cell inside : $g(t) = g$ constant concentration

outside : $h(t) = h$ constant concentration

Diffusion constant $D > 0$.

Find the steady state oxygene distribution inside the cell membrane:

Initial boundary value problem

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} \\ u(0,x) &= f(x) \\ u(t,0) = g \,&,\quad u(t,L) = h \end{aligned}\right\}$$

(a Dirichlet problem)

A steady state does not depend on time, hence

$$\frac{\partial u(t,x)}{\partial t} = 0 \quad , \quad u(t,x) = \bar{u}(x)$$

Then

$$\frac{\partial^2}{\partial x^2}\bar{u}(x) = 0 \qquad \text{or} \qquad \bar{u}'' = 0$$

$$\Rightarrow \qquad \bar{u}' = c_1 \qquad \Rightarrow \qquad \bar{u}(x) = c_1 x + c_2$$

Now, what is c_1 and c_2 ?

We have to use the boundary conditions:

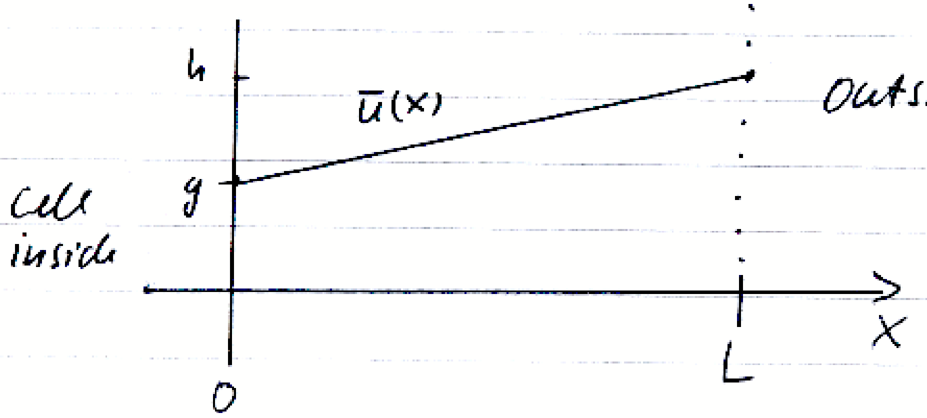
$$\bar{u}(0) = g = c_2 \qquad \Rightarrow \qquad c_2 = g$$

$$\bar{u}(L) = h = c_1 L + c_2 = c_1 L + g$$

$$\frac{h-g}{L} = c_1 \qquad \Rightarrow \qquad c_1 = \frac{h-g}{L}$$

We find $\boxed{\bar{u}(x) = \frac{h-g}{L}x + g}$

test: $\bar{u}(0) = g$
$\bar{u}(L) = h$



Steady state does not depend on $f(x)$ or on D!

An example for the Laplace operator in polar coordinates:

Find Δf, where $f(x,y) = x^2 + y^2$ or $f(r,\theta) = r^2$

Cartesian coordinates $\Delta f(x,y) = \frac{\partial^2}{\partial x^2}(x^2+y^2) + \frac{\partial^2}{\partial y^2}(x^2+y^2)$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y)$$

$$= 4$$

$$\Delta f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(f(r,\theta))\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} f(r,\theta)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r} r^2\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(2r^2\right) = \frac{1}{r}4r = 4$$

homogeneous

<u>Example 2</u> (The Neumann problem)

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) \\ &u(0,x) = f(x) \\ &\frac{\partial u}{\partial x}(t,0) = 0 , \quad \frac{\partial u}{\partial x}(t,L) = 0 \end{aligned}\right\}$$

Steady states: $\frac{\partial u(t,x)}{\partial t} = 0$, hence $\frac{\partial^2}{\partial x^2}\bar{u}(x) = 0$

or $\bar{u}'' = 0 \quad \Rightarrow \quad \bar{u}(x) = c_1 x + c_2$

Boundary conditions: $\frac{\partial \bar{u}}{\partial x}(x) = c_1$

$$\frac{\partial \bar{u}}{\partial x}(0) = 0 , \quad \frac{\partial \bar{u}}{\partial x}(L) = 0 \Rightarrow c_1 = 0$$

If we set $c_1 = 0$ we satisfy both boundary conditions.
How do we find c_2?
We need to use the initial condition $u(0,x) = f(x)$!

<u>Conservation of mass for the homogeneous Neumann problem</u>:

We integrate the diffusion equation over $[0, L]$:

$$\int_0^L \frac{\partial}{\partial t}u(t,x)\,dx = \int_0^L D\frac{\partial^2}{\partial x^2}u(t,x)\,dx = D\left[\frac{\partial}{\partial x}u(t,L) - \frac{\partial}{\partial x}u(t,0)\right]$$

$\parallel$ Leipniz $\qquad\qquad = 0$

$$\frac{\partial}{\partial t}\int_0^L u(t,x)\,dx$$

Hence $\frac{\partial}{\partial t} \int_0^L u(t,x)\,dx = 0$

$\Rightarrow \int_0^L u(t,x)\,dx = \text{const.}$ The total mass is conserved.

(the total number of bacteria is conserved,
the total heat is preserved)

Hence $\int_0^L \bar{u}(x)\,dx = \int_0^L u(t,x)\,dx = \int_0^L u(0,x)\,dx$

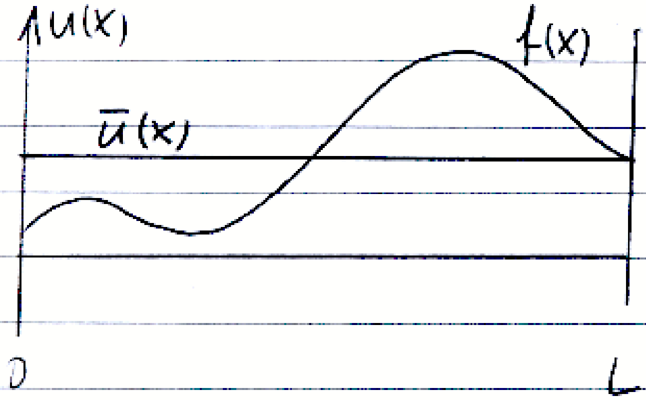
$= \int_0^L f(x)\,dx$

We use $\bar{u}(x) = c_2$, then

$\int_0^L \bar{u}(x)\,dx = c_2 L = \int_0^L f(x)\,dx \Rightarrow c_2 = \frac{1}{L}\int_0^L f(x)\,dx$

c_2 is the mean-value of $f(x)$ over $[0, L]$:

Steady state $\bar{u}(x) = \frac{1}{L}\int_0^L f(x)\,dx = \text{constant.}$



Source terms

We can add source or sink terms due to production of heat, or proliferation of bacteria, or chemical reactions (depending on the interpretation), $Q(t,x)$:

$$\boxed{\frac{\partial}{\partial t} u(t,x) = D \frac{\partial^2}{\partial x^2} u(t,x) + Q(t,x)}$$

Example 3: Find the steady state for

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial x^2} + 9\sin x \\ u(0,x) &= 9\sin x \\ u(t,0) &= 9 \\ \frac{\partial}{\partial x} u(t,2\pi) &= 0 \end{aligned}\right\}$$

Steady state: $\bar{u}(x)$: $3\bar{u}'' + 9\sin x = 0$

$$\bar{u}'' = -3\sin x$$

$$\Rightarrow \quad \bar{u}' = 3\cos x + C_1$$

$$\Rightarrow \quad \bar{u} = 3\sin x + C_1 x + C_2$$

$u(t,0) = 9$, $\quad \bar{u}(0) = C_2 = 9$

$\frac{\partial}{\partial x}\bar{u}(x) = 3\cos x + C_1$ $\qquad \frac{\partial}{\partial x}\bar{u}(2\pi) = 0$

$$+3 + C_1 = 0 \Rightarrow C_1 = -3$$

Steady state: $\boxed{\bar{u}(x) = 3\sin x - 3x + 9}$