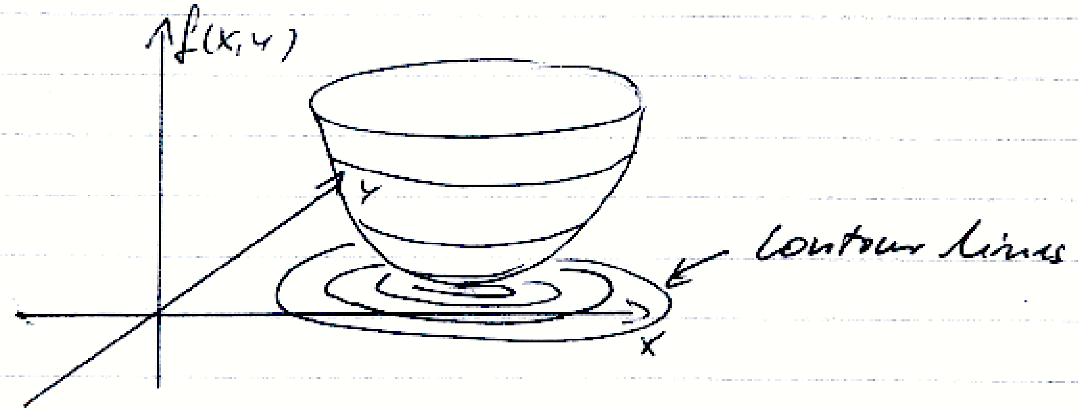
Introduction to partial differential equations

1. Introduction

(1.1) Partial derivatives

$D \subseteq \mathbb{R}^2$, $f : D \to \mathbb{R}$

$$\mathrm{graph}(f) = \{ ((x,y), f(x,y)) : (x,y) \in D \}$$



Example: $f(x,y) = \frac{1}{2}x^2 + y^2(1-y^2)$

The level sets, or contour lines are curves in the (x,y)-plane such that $f(x,y) = h$ constant along these curves

f depends on two variables, hence we can define two derivatives

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

which we call the <u>partial derivatives</u> of f.

gradient of f:

$$\operatorname{grad} f(x,y) = \begin{pmatrix} \frac{\partial}{\partial x} f(x,y) \\ \frac{\partial}{\partial y} f(x,y) \end{pmatrix}$$

<u>Example:</u> $f(x,y) = \frac{1}{2}x^2 + y^2(1-y^2)$

$$\frac{\partial}{\partial x} f(x,y) = x \qquad \frac{\partial}{\partial y} f(x,y) = 2y - 4y^3$$

$$\operatorname{grad} f(x,y) = \begin{pmatrix} x \\ 2y - 4y^3 \end{pmatrix}.$$

In 3-dimensions: $\operatorname{grad} f(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x} f(x,y,z) \\ \frac{\partial}{\partial y} f(x,y,z) \\ \frac{\partial}{\partial z} f(x,y,z) \end{pmatrix}$

<u>nabla</u> operator $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$

$\nabla f = \operatorname{grad} f$ If $g: \mathbb{R}^3 \to \mathbb{R}^3$ then

$$\nabla \cdot g = \operatorname{div} g = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}$$

<u>Laplace Operator</u> $\Delta = \nabla^2 = \nabla \cdot \nabla = \operatorname{div} \operatorname{grad}$

$$\Delta f(x,y,z) = \frac{\partial^2}{\partial x^2} f(x,y,z) + \frac{\partial^2}{\partial y^2} f(x,y,z) + \frac{\partial^2}{\partial z^2} f(x,y,z)$$

(1.2) Examples of partial differential equations

Def A *partial differential equation* (PDE) is an equation, or a system of equations, that involves partial derivatives.

Examples:

1) traveling wave: $\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = 0$, $f(t,x)$

   other notation: $f_t + c f_x = 0$, $f(t,x)$

2) wave equation: $\frac{\partial^2}{\partial t^2} f = \omega^2 \frac{\partial^2 f}{\partial x^2}$, $f_{tt} = D \Delta f$, $f(t,x)$

3) diffusion equation, or heat-equation

   $\frac{\partial}{\partial t} f = D \frac{\partial^2}{\partial x^2} f$ or $f_t = D \Delta f$, $f(t,x)$

4) Laplace equation $\frac{\partial^2}{\partial x^2} f = 0$, $\Delta f = 0$ $f(x)$

   or Poisson equation $\Delta f = \varphi(x)$.

5) Maxwell equations: $\operatorname{div} D = \rho$, $\nabla \times E = 0$

$\operatorname{div} B = 0$, $\nabla \times H = j$

6) Schrödinger equation: $-\frac{\hbar}{i}\frac{\partial}{\partial t} f + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} f = 0$ $\quad f(t,x)$

7) Reaction-diffusion equations for populations

$$\frac{\partial f}{\partial t} = D\Delta f + H(f) \qquad f(t,x)$$

8) Navier-Stokes equations $u_t - \nu \Delta u + (u\cdot\nabla u) + \nabla p = f$, $\nabla\cdot u = 0$

(1.3) Classification of linear 2nd order PDE's

A 2nd order PDE involves 2nd order derivatives.

Let $f(x,y)$ we study now:

$$a\frac{\partial^2}{\partial x^2} f(x,y) + 2b\frac{\partial^2}{\partial x \partial y} f(x,y) + c\frac{\partial^2}{\partial y^2} f(x,y) \qquad (1)$$

$$+ d\frac{\partial}{\partial x} f(x,y) + e\frac{\partial}{\partial y} f(x,y) + g f(x,y) = h$$

All terms which involve 2nd order derivatives are called "principal part" $a\frac{\partial^2}{\partial x^2} f + 2b\frac{\partial^2}{\partial x\partial y} f + c\frac{\partial^2}{\partial y^2} f$ (2)

To classify we study the corresponding characteristic equation

$$a\varphi^2 + 2b\varphi + c = 0$$

for an unknown parameter φ. Solve for φ:

$$\varphi_{1/2} = \frac{-b \pm \sqrt{b^2 - ac}}{a} \qquad \text{discriminant } b^2 - ac$$

If $b^2 - ac > 0$ we have two real roots, the equation is called <u>hyperbolic</u>.

If $b^2 - ac = 0$ we have a double real root: <u>parabolic</u>

If $b^2 - ac < 0$ we have complex roots: <u>elliptic</u>

<u>Definition</u> The linear second order PDE (1) is called

<u>hyperbolic</u>, if $b^2 - ac > 0$
<u>parabolic</u>, if $b^2 - ac = 0$
<u>elliptic</u>, if $b^2 - ac < 0$

<u>Examples:</u> 1) Laplace equation $\Delta f(x,y) = 0$

$$\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f = 0 \qquad a = 1,\ b = 0,\ c = 1$$

$$b^2 - ac = -1 < 0 \implies \underline{\text{elliptic}}$$

2) Heat equation $\frac{\partial}{\partial t} f(t,x) = D \frac{\partial^2}{\partial x^2} f(t,x)$ $\quad a = 0,\ b = 0,\ c = -D$

$$b^2 - ac = 0 \implies \underline{\text{parabolic}}$$

3) Wave equation: $\frac{\partial^2}{\partial t^2} f(t,x) = \omega^2 \frac{\partial^2}{\partial x^2} f(t,x)$

$a = 1,\ b = 0,\ c = -\omega^2$ $\qquad b^2 - ac = \omega^2 > 0$

$\Rightarrow$ <u>hyperbolic</u>

4) Tricomi's equation: $y \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = 0$

$a = y,\ b = 0,\ c = 1$

$b^2 - ac = -y,$

hyperbolic for $y < 0$
parabolic at $y = 0$
elliptic for $y > 0$.