MATH 300 Fall 2004
Advanced Boundary Value Problems I
Solutions to Assignment 1
Due: Friday September 24, 2004

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Question 1. [p 5, #8]

Derive the general solution of the equation

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = u, \quad a,\, b \neq 0$$

by using an appropriate change of variables.

SOLUTION: Let

$$\alpha = Ax + Bt \qquad \text{and} \qquad \beta = Cx + Dt,$$

where A, B, C, and D are to be determined so as to reduce the partial differential equation to and ordinary differential equation, which we can then solve.

From the chain rule, we have

$$\frac{\partial u}{\partial x} = A\frac{\partial u}{\partial \alpha} + C\frac{\partial u}{\partial \beta}$$
$$\frac{\partial u}{\partial t} = B\frac{\partial u}{\partial \alpha} + D\frac{\partial u}{\partial \beta}$$

and the original partial differential equation becomes

$$(aB + bA)\frac{\partial u}{\partial \alpha} + (aD + bC)\frac{\partial u}{\partial \beta} = u.$$

Now let $B = -b$, $A = a$, $C = 0$, and $D = 1/a$, then the equation becomes

$$\frac{\partial u}{\partial \beta} - u = 0,$$

and multiplying this equation by $e^{-\beta}$, we have

$$e^{-\beta}\frac{\partial u}{\partial \beta} - e^{-\beta}u = 0,$$

that is,

$$\frac{\partial}{\partial \beta}\left(e^{-\beta}u\right) = 0,$$

and the quantity $e^{-\beta}u$ is independent of α. Therefore, the solution is

$$u = f(\alpha)e^{\beta},$$

where f is an arbitrary function of α. In terms of the original variables, the solution is

$$u(x,t) = f(ax - bt)e^{t/a}.$$

Question 2. [p 14, #10]

Use d'Alembert's method and the superposition principle to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with initial data

$$u(x,0) = e^{-x^2}, \quad \frac{\partial u}{\partial t}(x,0) = \frac{x}{(1+x^2)^2}, \quad -\infty < x < \infty.$$

Solution: Using the change of variables

$$\alpha = x + ct \qquad \text{and} \qquad \beta = x - ct,$$

then from the chain rule we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta},$$

and replacing u by $\dfrac{\partial u}{\partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right),$$

that is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}$$

Again, from the chain rule, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta},$$

and replacing u by $\dfrac{\partial u}{\partial t}$, we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) = c\frac{\partial}{\partial \alpha}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) - c\frac{\partial}{\partial \beta}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right),$$

that is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2},$$

and substituting these expressions into the wave equation, we obtain

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

This equation says that $\dfrac{\partial u}{\partial \beta}$ doesn't depend on α, and therefore

$$\frac{\partial u}{\partial \beta} = g(\beta),$$

where g is an arbitrary differentiable function.

Now, integrating this equation with respect to β, holding α fixed, we get

$$u = \int \frac{\partial u}{\partial \beta}\, d\beta + F(\alpha) = \int g(\beta)\, d\beta + F(\alpha) = F(\alpha) + G(\beta),$$

where F is an arbitrary differentiable function and G is an antiderivative of g.

Finally, using the fact that $\alpha = x+ct$ and $\beta = x-ct$, we get **d'Alembert's solution** to the one-dimensional wave equation:

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are arbitrary differentiable functions.

Now, in order to solve the original question, we solve the following initial-boundary-value problems, and use the superposition principle to combine them to get a solution to the original problem:

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geq 0, \\ v(x,0) &= e^{-x^2}, \quad -\infty < x < \infty \\ \frac{\partial v}{\partial t}(x,0) &= 0 \quad -\infty < x < \infty, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= c^2 \frac{\partial^2 w}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geq 0, \\ w(x,0) &= 0, \quad -\infty < x < \infty \\ \frac{\partial w}{\partial t}(x,0) &= \frac{x}{(1+x^2)^2} \quad -\infty < x < \infty, \end{aligned} \tag{2}$$

the solution to the original problem is then $u = v + w$. (Check this!!!)

For problem (1), we use the initial conditions to write

$$v(x,0) = e^{-x^2} = F(x) + G(x),$$

so that $F(x) + G(x) = e^{-x^2}$, and

$$\frac{\partial v}{\partial t} = 0 = cF'(x) - cG'(x),$$

so that

$$F(x) - G(x) = C,$$

where C is an arbitrary constant. Therefore,

$$2F(x) = e^{-x^2} + C \qquad \text{and} \qquad 2G(x) = e^{-x^2} - C,$$

and the solution to the first problem is

$$v(x,t) = F(x+ct) + G(x-ct) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right].$$

For problem (2), we use the initial conditions to write

$$w(x,0) = 0 = F(x) + G(x),$$

so that $G(x) = -F(x)$, and

$$\frac{\partial w}{\partial t}(x,0) = \frac{x}{(1+x^2)^2} = cF'(x) - cG'(x),$$

so that $cF'(x) - cG'(x) = 2cF'(x) = \dfrac{x}{(1+x^2)^2}$, and integrating we have

$$2cF(x) = \frac{1}{2} \cdot \frac{-1}{1+x^2} + 2cC,$$

where C is an arbitrary constant. Therefore,

$$F(x) = \frac{-1}{4c(1+x^2)} + C \qquad \text{and} \qquad G(x) = \frac{1}{4c(1+x^2)} - C$$

and the solution to the second problem is

$$w(x,t) = \frac{1}{4c}\left[\frac{-1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right].$$

The solution to the original initial value boundary value problem is then

$$u(x,t) = v(x,t) + w(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] + \frac{1}{4c}\left[\frac{-1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right].$$

Question 3. [p 24, #16]

Suppose that f is T-periodic and let F be an antiderivative of f, that is,

$$F(x) = \int_a^x f(t)\,dt, \quad -\infty < x < \infty.$$

Show that F is T-periodic if and only if the integral of f over an interval of length T is 0.

Solution: Note that

$$F(x+T) = \int_a^{x+T} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+T} f(t)\,dt = F(x) + \int_x^{x+T} f(t)\,dt$$

for all $x \in \mathbb{R}$, and therefore $F(x+T) = F(x)$ for all $x \in \mathbb{R}$ if and only if

$$\int_x^{x+T} f(t)\,dt = 0$$

for all $x \in \mathbb{R}$, that is, if and only if the integral of f over *any* interval of length T is 0. Since f is T-periodic, then F is T-periodic if and only if

$$\int_0^T f(t)\,dt = 0.$$

Question 4. [p 25, #22]

Triangular Wave. Let $f(x) = x - 2\left[\frac{x+1}{2}\right]$, and consider the function

$$h(x) = |f(x)| = \left|x - 2\left[\tfrac{x+1}{2}\right]\right|.$$

(a) Show that h is 2-periodic.

(b) Plot the graph of h.

(c) Generalize (a) by finding a closed formula that describes the $2p$-periodic triangular wave

$$g(x) = |x| \quad \text{if} \quad -p < x < p,$$

and

$$g(x + 2p) = g(x) \quad \text{otherwise.}$$

Solution: Note that if we can show that

$$f(x) = x - 2\left[\tfrac{x+1}{2}\right]$$

is 2-periodic, then for any $x \in \mathbb{R}$, we have

$$h(x + 2) = |f(x + 2)| = |f(x)| = h(x)$$

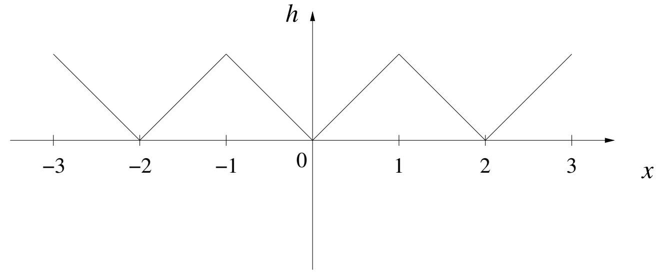
for all $x \in \mathbb{R}$, so that h is also 2-periodic.

(a) Now,

$$\begin{aligned} f(x+2) &= x + 2 - 2\left[\frac{(x+2)+1}{2}\right] \\ &= x + 2 - 2\left[\frac{x+1}{2} + 1\right] \\ &= x + 2 - 2\left(\left[\frac{x+1}{2}\right] + 1\right) \\ &= x - 2\left[\frac{x+1}{2}\right] \\ &= f(x) \end{aligned}$$

and f is 2-periodic, and from the remark above $h = |f|$ is also 2-periodic.

(b) Since $f(x) = x$ for $-1 < x < 1$, then $h(x) = |x|$ for $-1 < x < 1$, and the graph of h is shown below.



(c) In order to find a $2p$-periodic triangular wave, we use the $2p$-periodic function

$$f(x) = x - 2p\left[\frac{x+p}{2p}\right],$$

and note that $f(x) = x$ on the interval $-p < x < p$. We leave it to you to check, exactly as in part (a), that this is $2p$-periodic and that $f(x) = x$ for $-p < x < p$. Therefore,

$$g(x) = \left|x - 2p\left[\frac{x+p}{2p}\right]\right|$$

is a $2p$-periodic triangular wave which is equal to $|x|$ on the interval $-p < x < p$.

Question 5. [p 35, #6]

The function f is a 2π-periodic function and on the interval $-\pi \le x \le \pi$, we have

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2, \\ 0 & \text{if} \quad \pi/2 < |x| < \pi, \\ -1 & \text{if} \quad -\pi/2 < x < 0. \end{cases}$$

(a) Show that the Fourier series for f is given by $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin nx$.

(b) For which values of x does the Fourier series for f converge? Sketch the graph of the Fourier series.

SOLUTION:

(a) Note that f is an odd function on the interval $-\pi < x < \pi$, so that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0,$$

for $n = 1, 2, \dots$.

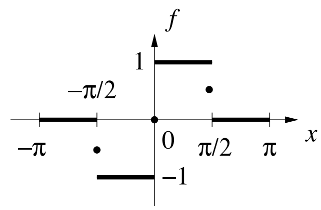
We use Euler's formula to calculate the b_n,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi/2}\sin nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi/2}\sin nx\,dx \\ &= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]\Bigg|_0^{\pi/2} \\ &= \frac{2}{n\pi}\left[1-\cos\frac{n\pi}{2}\right] \end{aligned}$$

and the Fourier series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin nx.$$

(b) The Fourier series of the graph of f on the interval $-\pi < x < \pi$ is shown below.



Note that the original function f is piecewise smooth and has only a finite jump discontinuity at $x = 0$ and $x = \pm\pi/2$, thus, from the Fourier Series Representation Theorem, the Fourier series of f will converge to 0 at all points $x = 2n\pi,\ n = 0, \pm1, \pm2, \pm3, \ldots$, and for all points $(2n+1)\pi/2$, the Fourier series of f will converge to $(-1)^n\pi/2,\ n = 0, \pm1, \pm2, \pm3, \ldots$. The rest of the graph of the Fourier series can be obtained by translating this graph by an integer multiple of 2π in the x-direction.

Question 6. [p 35, #8]

The function f is 2π-periodic and on the interval $-\pi \le x \le \pi$, we have $f(x) = |\cos x|$.

(a) Show that the Fourier series for f is given by $\dfrac{2}{\pi} - \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{(-1)^n}{4n^2-1}\cos 2nx$.

(b) For which values of x does the Fourier series for f converge? Sketch the graph of the Fourier series.

SOLUTION:

(a) Note that f an even function since

$$f(-x) = |\cos(-x)| = |\cos x| = f(x)$$

for all $x \in \mathbb{R}$, therefore $b_n = 0$ for all $n \ge 1$, and we only need to compute a_n for $n \ge 0$.

Now,

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\cos x|\,dx = \frac{1}{\pi}\int_{0}^{\pi} |\cos x|\,dx = \frac{1}{\pi}\int_{0}^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx \\
&= \frac{1}{\pi}\sin x\Big|_0^{\pi/2} - \frac{1}{\pi}\sin x\Big|_{\pi/2}^{\pi} = \frac{1}{\pi} - (-1)\frac{1}{\pi} = \frac{2}{\pi},
\end{aligned}$$

and for $n \ge 1$, since $\cos nx$ is also an even function, we have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\cos nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi/2} \cos x\cos nx\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos x\cos nx\,dx.
\end{aligned}$$

If $n = 1$, then

$$\begin{aligned}
a_1 &= \frac{2}{\pi}\int_{0}^{\pi/2} \cos^2 x\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos^2 x\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx \\
&= \frac{2}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = \frac{2}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right] = 0.
\end{aligned}$$

Now,

$$2\cos x \cos nx = \cos(n+1)x + \cos(n-1)x$$

so that for $n \neq 1$, we have

$$\frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi/2} (\cos(n+1)x + \cos(n-1)x)\, dx = \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right],$$

and

$$\frac{2}{\pi}\int_{\pi/2}^{\pi} \cos x \cos nx\, dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} (\cos(n+1)x + \cos(n-1)x)\, dx = -\frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right].$$

For $n \neq 1$, we have

$$a_n = \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right],$$

and if n is odd, then $a_n = 0$.

However, if n is even, say $n = 2k$, then

$$\begin{aligned} a_{2k} &= \frac{2}{\pi}\left[\frac{\sin(2k+1)\pi/2}{2k+1} + \frac{\sin(2k-1)\pi/2}{2k-1}\right] \\ &= \frac{2}{\pi}\left[\frac{(-1)^k}{2k+1} - \frac{(-1)^k}{2k-1}\right] \\ &= \frac{4}{\pi}\frac{(-1)^k}{4k^2-1}. \end{aligned}$$

and the Fourier series is

$$a_0 + \sum_{k=1}^{\infty} a_{2k}\cos 2kx = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos 2kx.$$

(b) Since $f(-\pi) = f(\pi)$, then the piecewise smooth 2π-periodic function with $f(x) = |\cos x|$, $-\pi \leq x \leq \pi$ is continuous at each $x \in \mathbb{R}$, and therefore the Fourier series converges to $f(x)$ for each $x \in \mathbb{R}$.

Question 7. [p 45, #4]

The function f is $2p$-periodic and is given on the interval $-p \leq x \leq p$ by $f(x) = x^2$. Show that the Fourier series of f is given by

$$\frac{p^2}{3} - \frac{4p^2}{\pi^2}\left[\cos(\pi x/p) - \frac{1}{2^2}\cos(2\pi x/p) + \frac{1}{3^2}\cos(3\pi x/p) - + \cdots\right]$$

and find its values at the points of discontinuity of f.

Solution: Note that since $f(p) = p^2 = (-p)^2 = f(-p)$, then the piecewise smooth $2p$-periodic function is continuous everywhere, and so has no points of discontinuity.

Also, since f is an even function, then $b_n = 0$ for all $n \geq 1$, and the Fourier series for f has only cosine terms:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/p)$$

where $a_0 = \dfrac{1}{p}\displaystyle\int_0^p f(x)\, dx$ and $a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos(n\pi x/p)\, dx$ for $n \geq 1$.

In order to calculate the coefficients a_n, we have

$$a_0 = \frac{1}{p}\int_0^p x^2\,dx = \frac{1}{p}\frac{x^3}{3}\bigg|_0^p = \frac{p^2}{3}.$$

For $n \geq 1$, we integrate by parts twice to get

$$\begin{aligned} a_n &= \frac{2}{p}\int_0^p x^2 \cos(n\pi x/p)\,dx = \frac{2}{p}\left[\frac{p}{n\pi}x^2 \sin(n\pi x/p)\bigg|_0^p - \frac{2p}{n\pi}\int_0^p x\sin(n\pi x/p)\,dx\right] \\ &= \frac{4}{n\pi}\int_0^p x\sin(n\pi x/p)\,dx = \frac{4}{n\pi}\left[-\frac{p}{n\pi}x\cos(n\pi x/p)\bigg|_0^p + \frac{p}{n\pi}\int_0^p \cos(n\pi x/p)\,dx\right] \\ &= \frac{4p^2}{n^2\pi^2}(-1)^n \end{aligned}$$

for $n = 1, 2, 3, \ldots$. The Fourier series of f is

$$\frac{p^2}{3} - \frac{4p^2}{\pi^2}\left[\cos(\pi x/p) - \frac{1}{2^2}\cos(2\pi x/p) + \frac{1}{3^2}\cos(3\pi x/p) - + \cdots\right],$$

and since f is piecewise smooth and continuous everywhere, the Fourier series given above converges to $f(x)$ for each $x \in \mathbb{R}$.

Question 8. [p 45, #28]

The function f is $2p$-periodic and is given on the interval $-p < x < p$ by $f(x) = x$. Show that the Fourier series of f is given by

$$\frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/p)$$

by differentiating the Fourier series in the previous problem term by term. Justify your work.

SOLUTION: Since the $2p$-periodic fuction $F(x)$ in the previous section is piecewise smooth and continuous everywhere, the Fourier series converges to the function everywhere, and

$$F(x) = \frac{p^2}{3} - \frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x/p)$$

where $F(x) = x^2$ for $-p < x < p$. Since this function also has a piecewise smooth derivative, and

$$F'(x) = 2x = 2 \cdot f(x)$$

for $-p < x < p$, then the coefficients in the Fourier series of $F'(x)$ can be obtained from Euler's formulas, or, they can be obtained by differentiating the above series term-by-term. Therefore, the Fourier series of $F'(x)$ is given by

$$\frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\pi}{n^2 p}\sin(n\pi x/p) = \frac{4p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/p),$$

and the Fourier series of $f(x)$ is

$$\frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/p),$$

which converges to $f(x)$ for all $x \neq \pm np$, and to 0 for $x = \pm np$.

Question 9. [p 66, #12]

Obtain the expansion

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

valid for all real numbers $a \neq 0$, and all $-\pi < x < \pi$.

SOLUTION: If $f(x)$ is a 2π-periodic piecewise smooth function, the complex form of the Fourier series of $f(x)$ is

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where the Fourier coefficients are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}\, dt.$$

Here the N^{th} partial sum

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

is the same as the usual partial sum (check this).

Now, if $f(x) = e^{ax}$ for $-\pi < x < \pi$, then

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx = \frac{1}{2\pi}\frac{e^{(a-in)x}}{a-in}\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\frac{e^{(a-in)\pi} - e^{-(a-in)\pi}}{a-in} = \frac{1}{2\pi}\frac{e^{a}(-1)^n - e^{-a}(-1)^n}{a-in} \\ &= \frac{(-1)^n \sinh \pi a}{\pi(a-in)} = \frac{(-1)^n(a+in)\sinh \pi a}{\pi(a^2+n^2)}, \end{aligned}$$

and the Fourier series of f is

$$\frac{\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{(a^2+n^2)}\,e^{inx} = \frac{\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx),$$

where we used the fact that

$$(a+in)e^{inx} = (a+in)(\cos nx + i\sin nx) = (a\cos nx - n\sin nx) + i(a\sin nx + n\cos nx),$$

and the fact that the Fourier series of a real valued function is real valued, so that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\sin nx + n\cos nx) = 0.$$

Since the function f is piecewise smooth and is continuous for $-\pi < x < \pi$, then we have

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

for $-\pi < x < \pi$.

Question 10.

Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity*:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\,\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Solution: If $z \neq 1$, then

$$\begin{aligned}(1-z)(1 + z + z^2 + \cdots z^n) &= 1 + z + z^2 + \cdots + z^n - (z + z^2 + \cdots + z^{n+1})\\ &= 1 - z^{n+1},\end{aligned}$$

so that

$$1 + z + z^2 + \cdots z^n = \begin{cases} \dfrac{1 - z^{n+1}}{1 - z} & \text{if } z \neq 1 \\ n + 1 & \text{if } z = 1. \end{cases}$$

Taking $z = e^{i\theta}$, where $0 < \theta < 2\pi$, then $z \neq 1$, so that

$$\begin{aligned}1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} &= \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}\\ &= \frac{1 - e^{(n+1)i\theta}}{-e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)}\\ &= \frac{-e^{-i\theta/2}\left(1 - e^{(n+1)i\theta}\right)}{2i\sin(\theta/2)}\\ &= \frac{i\left(e^{-i\theta/2} - e^{(n+\frac{1}{2})i\theta}\right)}{2\sin(\theta/2)}\\ &= \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)} + \frac{i}{2\sin(\theta/2)}\Big(\cos(\theta/2) - \cos\left(n+\tfrac{1}{2}\right)\theta\Big)\end{aligned}$$

Equating real and imaginary parts, we have

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)}$$

for $0 < \theta < 2\pi$, and as an added bonus,

$$\sin\theta + \sin 2\theta + \cdot + \sin n\theta = \frac{1}{2}\cot(\theta/2) - \frac{\cos\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)}$$

for $0 < \theta < 2\pi$.