MATH 300 Fall 2004
Advanced Boundary Value Problems I

Dirichlet's Integral

Department of Mathematical and Statistical Sciences
University of Alberta

In this note, we show that

$$\boxed{\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}}$$

as follows. For each $n \geq 1$, define

$$u_n = \int_0^{\frac{\pi}{2}} \sin 2nx \cot x\,dx \qquad \text{and} \qquad v_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x}\,dx$$

(a) First we show that $u_{n+1} = u_n = \dfrac{\pi}{2}$ for all $n \geq 1$.

(b) Next we show that $v = \lim\limits_{n\to\infty} v_n = \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

(c) Finally, we integrate by parts and use L'Hospital's rule to show that $\lim\limits_{n\to\infty}(u_n - v_n) = 0$.

(d) From all of the above, we have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

SOLUTION:
(a)

$$\begin{aligned}
u_{n+1} - u_n &= \int_0^{\frac{\pi}{2}} \left[\sin 2(n+1)x - \sin 2nx\right]\cot x\,dx = 2\int_0^{\frac{\pi}{2}} \sin x\cos(2n+1)x\cot x\,dx \\
&= 2\int_0^{\frac{\pi}{2}} \cos x\cos(2n+1)x\,dx = \int_0^{\frac{\pi}{2}} \left[\cos(2n+2)x + \cos 2nx\right]\,dx \\
&= \frac{1}{2(n+1)}\sin 2(n+1)x\Big|_0^{\frac{\pi}{2}} + \frac{1}{2n}\sin 2nx\Big|_0^{\frac{\pi}{2}} = 0,
\end{aligned}$$

therefore $u_{n+1} - u_n = 0$ for all $n \geq 1$.
Now,

$$\begin{aligned}
u_1 &= \int_0^{\frac{\pi}{2}} \sin 2x\cot x\,dx = 2\int_0^{\frac{\pi}{2}} \cos^2 x\,dx \\
&= \int_0^{\frac{\pi}{2}} (1 + \cos 2x)\,dx = \frac{\pi}{2} + \frac{\sin 2x}{2}\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2},
\end{aligned}$$

and therefore $u_{n+1} = u_n = \dfrac{\pi}{2}$ for all $n \geq 1$.

(b) Make the substitution $t = 2nx$ in the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x}\,dx$, then

$$v_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x}\,dx = \int_0^{n\pi} \frac{\sin t}{t}\,dt,$$

and therefore

$$v = \lim_{n\to\infty} v_n = \int_0^{\infty} \frac{\sin t}{t}\,dt.$$

(c) Integrating by parts, we obtain

$$\begin{aligned} u_n - v_n &= \int_0^{\frac{\pi}{2}} \left(\cot x - \frac{1}{x}\right) \sin 2nx\,dx \\ &= -\frac{\cos 2nx}{2n}\left(\cot x - \frac{1}{x}\right)\Bigg|_0^{\frac{\pi}{2}} - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \left(\csc^2 x - \frac{1}{x^2}\right)\cos 2nx\,dx, \end{aligned}$$

and therefore

$$u_n - v_n = \frac{(-1)^n}{n\pi} + \lim_{p\to 0^+} \frac{\cos 2np}{2n}\left(\cot p - \frac{1}{p}\right) - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \left(\csc^2 x - \frac{1}{x^2}\right)\cos 2nx\,dx. \qquad (*)$$

Using L'Hospital's rule, we have

$$\lim_{p\to 0^+} \left(\cot p - \frac{1}{p}\right) = 0,$$

and

$$\lim_{x\to 0^+} \left(\csc^2 x - \frac{1}{x^2}\right) = \frac{1}{3}.$$

Therefore, we may redefine the integrand on the right-hand side of $(*)$ to be continuous on the interval $[0, \frac{\pi}{2}]$, and hence bounded there. Thus, there exists a constant $M > 0$ such that

$$\begin{aligned} \left|\int_0^{\frac{\pi}{2}} \left(\csc^2 x - \frac{1}{x^2}\right)\cos 2nx\,dx\right| &\le \int_0^{\frac{\pi}{2}} \left|\left(\csc^2 x - \frac{1}{x^2}\right)\cos 2nx\right| dx \\ &\le \int_0^{\frac{\pi}{2}} \left|\csc^2 x - \frac{1}{x^2}\right| dx \\ &\le \frac{\pi M}{2}. \end{aligned}$$

Letting $n \to \infty$ in $(*)$, we obtain

$$\lim_{n\to\infty} (u_n - v_n) = 0,$$

and therefore,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (u_n - v_n + v_n) = \lim_{n\to\infty} (u_n - v_n) + \lim_{n\to\infty} v_n,$$

so that

$$\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

Next we show that

$$\boxed{\int_0^\infty \left|\frac{\sin t}{t}\right| dt = +\infty}$$

as follows.

Let n be a positive integer,

(a) First we show that

$$\int_0^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{k=0}^{n} \int_0^\pi \frac{\sin x}{x+k\pi}\, dx.$$

(b) Next we show that

$$\int_0^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{\pi}\sum_{k=0}^{n} \frac{1}{k+1}.$$

(c) Finally, we show that

$$\sum_{k=0}^{\infty} \frac{1}{k+1} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k+1} = +\infty.$$

(d) From all of the above, we have

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx = +\infty.$$

Solution:

(a)

$$\begin{aligned}
\int_0^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx &= \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{k=0}^{n} \int_0^\pi \left|\frac{\sin(t+k\pi)}{t+k\pi}\right| dt \\
&= \sum_{k=0}^{n} \int_0^\pi \left|\frac{(-1)^k \sin t}{t+k\pi}\right| dt = \sum_{k=0}^{n} \int_0^\pi \frac{|\sin t|}{t+k\pi}\, dt \\
&= \sum_{k=0}^{n} \int_0^\pi \frac{\sin t}{t+k\pi}\, dt,
\end{aligned}$$

where the last equality follows since $\sin t \geq 0$ for $0 \leq t \leq \pi$.

(b) For $0 < t < \pi$, we have $\dfrac{1}{t+k\pi} > \dfrac{1}{\pi(k+1)}$, so that (a) implies that

$$\int_0^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx > \sum_{k=0}^{n} \frac{1}{\pi(k+1)} \int_0^\pi \sin t\, dt = \frac{2}{\pi}\sum_{k=0}^{n} \frac{1}{k+1}.$$

(c) For $t \geq 1$,

$$\log x = \int_1^x \frac{1}{t}\, dt \leq \int_1^x 1\, dt = x - 1$$

for $x \geq 1$, and replacing x by $\frac{k+2}{k+1}$, we have

$$\log\left(\frac{k+2}{k+1}\right) \leq \frac{1}{k+1}$$

for all $k \geq 0$. Therefore,

$$\sum_{k=0}^{n} \frac{1}{k+1} \geq \sum_{k=0}^{n} (\log(k+2) - \log(k+1)) = \log(n+2),$$

and so

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k+1} \geq \lim_{n\to\infty} \log(n+2) = +\infty.$$

(d)

$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx = \lim_{n\to\infty} \int_0^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx \geq \lim_{n\to\infty} \frac{2}{\pi} \sum_{k=0}^{n} \frac{1}{k+1}$$

$$\geq \lim_{n\to\infty} \frac{2}{\pi} \log(n+2) = +\infty,$$

Therefore,

$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx = +\infty.$$