MATH 300 Fall 2004
Advanced Boundary Value Problems I
Assignment 5
Due: Monday December 6, 2004

Department of Mathematical and Statistical Sciences
University of Alberta

Question 1. [p 395, #3]

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 - \cos x & \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Question 2. [p 395, #9]

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} x & \text{if} \quad -1 < x < 1, \\ 2 - x & \text{if} \quad 1 < x < 2, \\ -2 - x & \text{if} \quad 2 < x < -1, \\ 0 & \text{otherwise.} \end{cases}$$

Question 3. [p 407, #4]

Let

$$f(x) = \begin{cases} x & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Plot the function $f(x)$ and find its Fourier transform.

(b) If $\widehat{f}$ is real valued, plot it; otherwise plot $|\widehat{f}|$.

Question 4. [p 407, #10] Reciprocity relation for the Fourier transform.

(a) From the definition of transforms, explain why

$$\mathcal{F}(f)(x) = \mathcal{F}^{-1}(f)(-x).$$

(b) Use (a) to derive the **reciprocity relation**

$$\mathcal{F}^2(f)(x) = f(-x),$$

where $\mathcal{F}^2(f)(x) = \mathcal{F}(\mathcal{F}(f))$.

(c) Conclude the following: f is even if and only if $\mathcal{F}^2(f)(x) = f(x)$;
f is odd if and only if $\mathcal{F}^2(f)(x) = -f(x)$.

(d) Show that for any f, $\mathcal{F}^4(f) = f$.

Question 5. [p 410, #55] Basic Properties of Convolutions.

Establish the following properties of convolutions. (These properties can be derived directly from the definitions or by using the operational properties of the Fourier transform.)

(a) $f * g = g * f$ (commutativity).

(b) $f * (g * h) = (f * g) * h$ (associativity).

(c) Let a be a real number and let f_a denote the translate of f by a, that is,

$$f_a(x) = f(x - a).$$

Show that

$$(f_a) * g = f * (g_a) = (f * g)_a.$$

This important property says that convolutions commute with translations.

Question 6. [p 418, #3]

Determine the solution of the following initial boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = e^{-x^2}, \qquad -\infty < x < \infty.$$

Give your answer in the form of an inverse Fourier transform.

Question 7. [p 418, #11]

Solve the following initial boundary value problem

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty.$$

Assume that the function f has a Fourier transform.

Question 8. [p 426, #2]

Use convolutions, the error function, and operational properties of the Fourier transform to solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{100}\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = \begin{cases} 100 & \text{if} \quad -2 < x < 0, \\ 50 & \text{if} \quad 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Question 9. [p 439, #6]

Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1-x & \text{if} \quad 0 < x < 1, \\ 0 & \text{if} \quad x \geq 1. \end{cases}$$

and write $f(x)$ as an inverse cosine transform. Use a known Fourier transform and the fact that if $f(x),\ x \geq 0$, is the restriction of an *even* function f_e, then

$$\mathcal{F}_c(f)(\omega) = \mathcal{F}(f_e)(\omega)$$

for all $\omega \geq 0$.

Question 10. [p 439, #12]

Find the Fourier sine transform of

$$f(x) = \frac{x}{1+x^2}, \qquad x > 0,$$

and write $f(x)$ as an inverse sine transform. Use a known Fourier transform and the fact that if $f(x),\ x \geq 0$, is the restriction of an *odd* function f_o, then

$$\mathcal{F}_s(f)(\omega) = i\mathcal{F}(f_o)(\omega)$$

for all $\omega \geq 0$.