MATH 300 Fall 2004
Advanced Boundary Value Problems I
Assignment 2
Due: Friday October 8, 2004

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Question 1. **[p 77, #26]**

Solve the initial value problem

$$y'' + 9y = F(t)$$
$$y(0) = 0$$
$$y'(0) = 0$$

where $F(t)$ is the 2π-periodic input function given by its Fourier series $\quad F(t) = \sum_{n=1}^{\infty} \left[\frac{\cos nt}{n^2} + (-1)^n \frac{\sin nt}{n} \right].$

Question 2. **[p 107, #8]**

Verify that the function

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is a solution to the three dimensional Laplace equation $\quad u_{xx} + u_{yy} + u_{zz} = 0.$

Question 3. **[p 123, #2]**

Solve the one dimensional wave equation with the boundary conditions and initial conditions as given below

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{1}{\pi^2} \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(1,t) &= 0, \quad t > 0 \\ u(x,0) &= \sin \pi x \cos \pi x, \quad 0 < x < 1, \\ \frac{\partial u}{\partial t}(x,0) &= 0, \quad 0 < x < 1, \end{aligned}$$

using the Method of Separation of Variables.

Question 4. **[p 123, #4]**

Solve the one dimensional wave equation with the boundary conditions and initial conditions as given below

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(1,t) &= 0, \quad t > 0 \\ u(x,0) &= \sin \pi x + \tfrac{1}{2} \sin 3\pi x + 3 \sin 7\pi x, \quad 0 < x < 1, \\ \frac{\partial u}{\partial t}(x,0) &= \sin 2\pi x, \quad 0 < x < 1, \end{aligned}$$

using the Method of Separation of Variables.

Question 5. [p 124, #12]

Damped vibrations of a string. In the presence of resistance proportional to velocity, the one dimensional wave equation becomes

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad 0 < x < L,\ t > 0.$$

Solve this equation subject to the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \qquad \text{for all} \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad \text{for} \quad 0 < x < L$$

as follows:

(a) Assume a product solution of the form $u(x,t) = X(x)T(t)$, and derive the following equations for X and T,

$$X'' + \mu^2 X = 0, \quad X(0) = 0, \quad X(L) = 0,$$
$$T'' + 2kT' + (\mu c)^2 T = 0,$$

where μ is the separation constant.

(b) Show that $\mu = \mu_n = \dfrac{n\pi}{L}$ and $X = X_n = \sin(n\pi x/Li)$, $n = 1, 2, \ldots$.

(c) Solve the equation $T_n'' + 2kT_n' + (n\pi c/L)^2 T_n = 0$ for T_n, $n = 1, 2, \ldots$.

(d) Conclude that when $\frac{kL}{\pi c}$ is not a positive integer, the solution is

$$\begin{aligned} u(x,t) &= e^{-kt} \sum_{1 \le n < kL/\pi c} \sin(n\pi x/L)\left(a_n \cosh\lambda_n t + b_n \sinh\lambda_n t\right) \\ &\quad + e^{-kt} \sum_{kL/\pi c < n < \infty} \sin(n\pi x/L)\left(a_n \cos\lambda_n t + b_n \sin\lambda_n t\right) \end{aligned}$$

where these sums run over integers only, $\lambda_n = \sqrt{\left|k^2 - (n\pi c/L)^2\right|}$, and where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\ dx, \quad n = 1, 2, \ldots,$$

and the b_n are determined from the equation

$$-ka_n + \lambda_n b_n = \frac{2}{L}\int_0^L g(x)\sin(n\pi x/L)\ dx, \quad n = 1, 2, \ldots.$$

(e) Conclude that when $\frac{kL}{\pi c}$ is a positive integer, the solution is as in (d) with the one additional term

$$\sin(kx/c)\,(a_{kL/\pi c}e^{-kt} + b_{kL/\pi c}te^{-kt}$$

with a_n and b_n as in (d), except that $b_{kL/\pi c}$ is determined from the equation

$$-ka_{kL/\pi c} + b_{kL/\pi c} = \frac{2}{L}\int_0^L g(x)\sin(kx/c)\ dx.$$

Question 6. [p 133, #4]

Use D'Alembert's solution to solve the boundary value problem for the wave equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\
u(0,t) &= 0, \quad t > 0 \\
u(1,t) &= 0, \quad t > 0 \\
u(x,0) &= 0, \quad 0 < x < 1, \\
\frac{\partial u}{\partial t}(x,0) &= 1, \quad 0 < x < 1.
\end{aligned}$$

Question 7. [p 133, #8]

Use D'Alembert's solution to solve the boundary value problem for the wave equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\
u(0,t) &= 0, \quad t > 0 \\
u(1,t) &= 0, \quad t > 0 \\
u(x,0) &= 0, \quad 0 < x < 1, \\
\frac{\partial u}{\partial t}(x,0) &= \sin \pi x, \quad 0 < x < 1.
\end{aligned}$$

Question 8. [p 134, #16]

D'Alembert's solution for zero initial velocity. Show that the solution to the wave equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\
u(0,t) &= 0, \quad t > 0 \\
u(L,t) &= 0, \quad t > 0 \\
u(x,0) &= f(x), \quad 0 < x < L, \\
\frac{\partial u}{\partial t}(x,0) &= 0, \quad 0 < x < L
\end{aligned}$$

is given by

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \left[\sin\left(n\pi(x-ct)/L\right) + \sin\left(n\pi(x+ct)/L\right)\right]$$

where $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(n\pi x/L\right)\, dx,\ n = 1, 2, \ldots.$

Question 9. [p 144, #2]

Solve the boundary value problem for the one dimensional heat equation

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0 \\
u(0,t) &= 0, \quad t > 0 \\
u(\pi,t) &= 0, \quad t > 0 \\
u(x,0) &= 30\sin x, \quad 0 < x < \pi,
\end{aligned}$$

and give a brief physical explanation of the problem.

Question 10. [p 144, #6]

Solve the boundary value problem for the one dimensional heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(1,t) &= 0, \quad t > 0 \\ u(x,0) &= e^{-x}, \quad 0 < x < 1, \end{aligned}$$

and give a brief physical explanation of the problem.