MATH 300 Fall 2004
Advanced Boundary Value Problems I
Assignment 1
Due: Friday September 24, 2004

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Question 1. [p 5, #8]

Derive the general solution of the equation

$$a\frac{\partial u}{\partial t}+b\frac{\partial u}{\partial x}=u, \quad a, b\neq 0$$

by using an appropriate change of variables.

Question 2. [p 14, #10]

Use D'Alembert's method and the superposition principle to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

with initial data

$$u(x,0)=e^{-x^2}, \quad \frac{\partial u}{\partial t}(x,0)=\frac{x}{(1+x^2)^2}, \quad -\infty<x<\infty.$$

Question 3. [p 24, #16]

Suppose that f is T-periodic and let F be an antiderivative of f, that is,

$$F(x)=\int_a^x f(t)\,dt, \quad -\infty<x<\infty.$$

Show that F is T-periodic if and only if the integral of f over an interval of length T is 0.

Question 4. [p 25, #22]

Triangular Wave. Let $f(x)=x-2\left[\frac{x+1}{2}\right]$, and consider the function

$$h(x)=|f(x)|=\left|x-2\left[\tfrac{x+1}{2}\right]\right|.$$

(a) Show that h is 2-periodic.

(b) Plot the graph of h.

(c) Generalize (a) by finding a closed formula that describes the 2-periodic triangular wave

$$g(x)=|x| \quad \text{if} \quad -p<x<p,$$

and

$$g(x+2p)=g(x) \quad \text{otherwise.}$$

Question 5. [p 35, #6]

The function f is a 2π-periodic function and on the interval $-\pi \le x \le \pi$, we have

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2, \\ 0 & \text{if} \quad \pi/2 < |x| < \pi, \\ -1 & \text{if} \quad -\pi/2 < x < 0. \end{cases}$$

(a) Show that the Fourier series for f is given by $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos \frac{n\pi}{2}\right) \sin nx$.

(b) For which values of x does the Fourier series for f converge? Sketch the graph of the Fourier series.

Question 6. [p 35, #8]

The function f is 2π-periodic and on the interval $-\pi \le x \le \pi$, we have $f(x) = |\cos x|$.

(a) Show that the Fourier series for f is given by $\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \cos 2nx$.

(b) For which values of x does the Fourier series for f converge? Sketch the graph of the Fourier series.

Question 7. [p 45, #4]

The function f is $2p$-periodic and is given on the interval $-p \le x \le p$ by $f(x) = x^2$. Show that the Fourier series of f is given by

$$\frac{p^2}{3} - \frac{4p^2}{\pi^2} \left[\cos(\pi x/p) - \frac{1}{2^2} \cos(2\pi x/p) + \frac{1}{3^2} \cos(3\pi x/p) - + \cdots \right]$$

and find its values at the points of discontinuity of f.

Question 8. [p 45, #28]

The function f is $2p$-periodic and is given on the interval $-p < x < p$ by $f(x) = x$. Show that the Fourier series of f is given by

$$\frac{2p}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x/p)$$

by differentiating the Fourier series in the previous problem term by term. Justify your work.

Question 9. [p 66, #12]

Obtain the expansion

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} (a \cos nx - n \sin nx)$$

valid for all real numbers $a \neq 0$, and all $-\pi < v < \pi$.

Question 10.

Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity*:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2 \sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$