(10.5) <u>Example 3</u>

$$\begin{cases} \dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x^2} - \sin(2x) \\ u(0,t) = 1, \quad u(2\pi, t) = 2 \\ u(x,0) = x \end{cases}$$

We assume that $u(x,t) = v(x) + w(x,t)$,

if so, then

$$\begin{cases} \dfrac{\partial w}{\partial t} = 3v'' + 3\dfrac{\partial^2 w}{\partial x^2} - \sin(2x) \\ v(0) + w(0,t) = 1, \quad v(2\pi) + w(2\pi, t) = 2 \\ v(x) + w(x,0) = x \end{cases}$$

$\Rightarrow$ the v-problem $\begin{cases} 3v'' = \sin(2x) \\ v(0) = 1, \quad v(2\pi) = 2 \end{cases}$

the w-problem

$$\begin{cases} \dfrac{\partial w}{\partial t} = 3\dfrac{\partial^2 w}{\partial x^2} \\ w(0,t) = 0, \quad w(2\pi, t) = 0 \\ w(x,0) = x - v(x) \end{cases}$$

<u>The v-problem:</u>

$$v'' = \frac{\sin(2x)}{3}, \quad v' = \frac{-\cos(2x)}{6} + C_1$$

$$v(x) = \frac{-\sin(2x)}{12} + C_1 x + C_2$$

$$1 = v(0) = C_2$$

$$2 = v(2\pi) = + C_1 2\pi + 1 \implies C_1 = \frac{1}{2\pi}$$

$$\implies \quad v(x) = 1 + \frac{1}{2\pi} x - \frac{\sin(2x)}{12}$$

The w-problem:

$$\begin{cases} \dfrac{\partial w}{\partial t} = 3\dfrac{\partial^2 w}{\partial x^2}, & 0 < x < 2\pi \\ w(0,t) = 0, \quad w(2\pi, t) = 0 & \\ w(x,0) = x - 1 - \dfrac{x}{2\pi} + \dfrac{\sin(2x)}{12} & \\ \qquad = \left(1 - \dfrac{1}{2\pi}\right)x - 1 + \dfrac{\sin(2x)}{12} & \end{cases}$$

We need the Fourier-sin-series of $\left(1-\frac{1}{2\pi}\right)x - 1$:

$$\left(1-\frac{1}{2\pi}\right)x - 1 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n}{2}x\right)$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} \left[\left(1-\frac{1}{2\pi}\right)x - 1\right] \sin\left(\frac{n}{2}x\right) dx$$

$$= \frac{1}{\pi}\left(1-\frac{1}{2\pi}\right)\left[x \frac{-\cos\left(\frac{n}{2}x\right)}{n/2}\Big|_0^{2\pi} - \int_0^{2\pi} \frac{-\cos\left(\frac{n}{2}x\right)}{n/2}\, dx \right]$$

$$- \frac{1}{\pi}\,\frac{2}{n}\left(-\cos\left(\frac{n}{2}x\right)\right)\Big|_0^{2\pi}$$

$$= \frac{1}{\pi}\left(1-\frac{\cancel{1}}{2\pi}\right)\left[\frac{-2\pi\cos(n\cancel{\pi})\cdot 2}{n} + \underbrace{\frac{4}{n^2}\sin\left(\frac{n}{2}x\right)\Big|_0^{2\pi}}_{=0}\right]$$

$$+ \frac{2}{n\pi}\left(\cos(n\pi)-1\right)$$

$$= -\frac{4}{n}\left(1-\frac{1}{2\pi}\right)(-1)^n + \frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{4}{n}\left(\left(-1+\frac{1}{\pi}\right)(-1)^n - \frac{1}{2\pi}\right)$$

$$b_1 = 4\left(\frac{-\pi+1}{\pi}(-1) - \frac{1}{2\pi}\right) = 4\left(1-\frac{3}{2\pi}\right)$$

$$\left(1-\frac{1}{2\pi}\right)x-1=\sum_{n=1}^{\infty}\frac{4}{n}\left(\left(-1+\frac{1}{\pi}\right)(-1)^n-\frac{1}{2\pi}\right)\sin\left(\frac{n}{2}x\right)$$

$$b_2=2\left(-1+\frac{1}{2\pi}\right)$$

$$b_3=\frac{4}{3}\left(1-\frac{3}{2\pi}\right)$$

$$b_4=1\cdot\left(-1+\frac{1}{2\pi}\right)$$

Then

$$w(x,0)=\sum_{n=1}^{\infty}\frac{4}{n}\left(\left(-1+\frac{1}{\pi}\right)(-1)^n-\frac{1}{2\pi}\right)\sin\left(\frac{n}{2}x\right)+\frac{1}{12}\sin(2x)$$

$$=4\left(1-\frac{3}{2\pi}\right)\sin\left(\frac{x}{2}\right)+2\left(\frac{1}{2\pi}-1\right)\sin(x)+\frac{4}{3}\left(1-\frac{3}{2\pi}\right)\sin\left(\frac{3}{2}x\right)$$

$$+\left(\frac{1}{2\pi}-1\right)\sin(2x) + \sum_{n=5}^{\infty} b_n \sin\left(\frac{n}{2}x\right) + \frac{1}{12}\sin(2x)$$

$$\left(\frac{1}{2\pi}-\frac{11}{12}\right)\sin(2x).$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} B_n e^{-3\left(\frac{n^2}{4}t\right)} \sin\left(\frac{n}{2}x\right)$$

The u-problem: $u(x,t) = v(x) + w(x,t)$

$$= 1 + \frac{1}{2\pi}x - \frac{\sin(2x)}{12} + \sum_{n=1}^{\infty} B_n e^{-3\frac{n^2}{4}t} \sin\left(\frac{n}{2}x\right)$$

$$= 1 + \frac{1}{2\pi}x - \frac{\sin(2x)}{12} + 4\left(1-\frac{3}{2\pi}\right)e^{-\frac{3}{4}t}\sin\left(\frac{x}{2}\right) + 2\left(\frac{1}{2\pi}-1\right)e^{-3t}\sin(x)$$

$$+ \frac{4}{3}\left(1-\frac{3}{2\pi}\right)\sin\left(\frac{3}{2}x\right)e^{-\frac{27}{4}t} + \left(\frac{1}{2\pi}-\frac{11}{12}\right)\sin(2x)e^{-12t} + \cdots$$