Laplace Equation in Spherical Coordinates

$\nabla^2 u = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

Assume independence of ϕ: $\frac{\partial u}{\partial \phi} = 0$

$r \to 0$: $|u(r,\theta)| < \infty$

$r = a$: $u(a,\theta) = f(\theta)$

<u>Separation</u> $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\frac{1}{r^2}(r^2 R')'\,\Theta + \frac{R}{r^2 \sin\theta}(\sin\theta\,\Theta')' = 0 \qquad \Big|\cdot r^2$$

$$\frac{(r^2 R')'}{R} = -\frac{(\sin\theta\,\Theta')'}{\Theta \sin\theta} = \lambda$$

radial problem: $(r^2 R')' = \lambda R$

$|R(0)| < \infty$, $\boxed{R}$

angular problem: $(\sin\theta\,\Theta')' = -\lambda\Theta \sin\theta$

both are Sturm-Liouville problems.

The R-problem is an Eulers equation and is solved by $R_n(r) = C_1 r^n + \frac{C_2}{r^{n+1}}$

and eigenvalue $\lambda_n = n(n+1)$.

Since $|R(0)| < \infty$ we must have $C_2 = 0$:

$$R_n(r) = C_n r^n$$

θ-equation with $\lambda = n(n+1)$:

$$(\sin\theta \, \Theta')' = -\lambda \Theta \sin\theta = -n(n+1) \sin\theta \, \Theta$$

Introduce $\omega = \cos\theta$

then $\dfrac{d\Theta}{d\theta} = \dfrac{d\Theta}{d\omega} \cdot \dfrac{d\omega}{d\theta} = -\sin\theta \dfrac{d\Theta}{d\omega}$

Thus

$$(\sin\theta \, \Theta')' = \frac{d}{d\omega}\left(\sin\theta \frac{d\Theta}{d\omega}(-\sin\theta)\right)(-\sin\theta)$$

$$= -\sin\theta \frac{d}{d\omega}\left(-\sin^2\theta \frac{d\Theta}{d\omega}\right)$$

$$= -\sin\theta \frac{d}{d\omega}\left((\omega^2 - 1)\frac{d\Theta}{d\omega}\right)$$

$$= -n(n+1) \sin\theta \, \Theta$$

Hence

$$\frac{d}{d\omega}\left((\omega^2 - 1)\frac{d\Theta}{d\omega}\right) - n(n+1)\Theta = 0$$

Writing $'$ instead of $\dfrac{d}{d\omega}$:

$$((1-\omega^2)\,\Theta')' + n(n+1)\,\Theta = 0$$

which is Legendre's equation (SL-problem.

Solution $\Theta(\omega) = A_n P_n(\omega) + B_n Q_n(\omega)$

$P_n(\omega)$: Legendre polynomials (p. 337)

$Q_n(\omega)$: Legendre polynomials of second kind (p. 338).

The $Q_n(\omega)$ are not defined for $\omega = \pm 1$ hence $\cos\theta = \pm 1$. So we must have $B_n = 0$.

$$\Rightarrow \quad \Theta(\omega) = A_n P_n(\omega)$$

or $\Theta(\cos\theta) = A_n P_n(\cos\theta)$

Combining: $u(r,\theta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta)$

boundary condition at $r = a$:

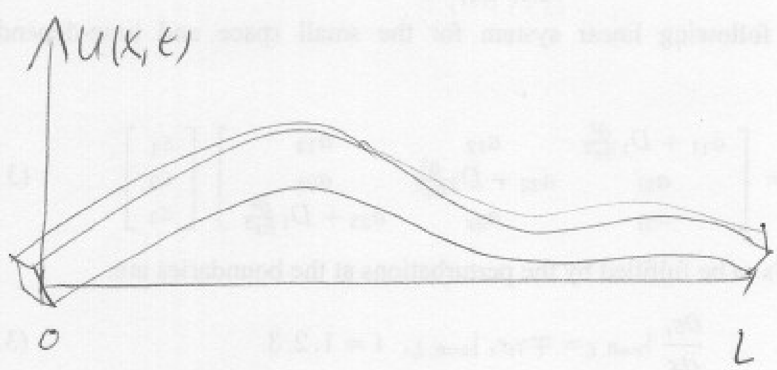
$$u(a,\theta) = f(\theta) = \sum_{n=0}^{\infty} C_n a^n P_n(\cos\theta)$$

with $$C_n = \frac{1}{a^n} \frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta$$

where we used $$\int_0^{\pi} P_n^2(\cos\theta) \sin\theta \, d\theta = \frac{2}{2n+1}$$

and $\sin\theta$ is the weight function of the θ- Sturm-Liouville problem.

(10.3) The Thin Beam Equations



Thin beam equation $u_{tt} + c^2 u_{xxxx} = 0$

(Derivation in the Textbook p. 364-367).

Simply supported: $u(0,t)=0, \quad u(L,t)=0$

$$u_{xx}(0,t)=0, \quad u_{xx}(L,t)=0$$

initial conditions: $u(x,0)=f(x)$

$$u_t(x,0)=g(x)$$

Separation: $u(x,t) = \varphi(x)\, G(t)$

$$\varphi G'' + c^2 G \varphi^{IV} = 0$$

$$\frac{G''}{c^2 G} = -\frac{\varphi^{IV}}{\varphi} = -\lambda$$

time problem $\quad G'' = -\lambda c^2 G \qquad (1)$

spatial problem
$$\left.\begin{aligned} &\varphi^{IV} = \lambda \varphi \\ &\varphi(0)=0, \quad \varphi(L)=0 \\ &\varphi''(0)=0, \quad \varphi''(L)=0 \end{aligned}\right\} \qquad (2)$$

(2) is a fourth order ODE and we expect 4 linear independent solutions. The characteristic equation reads

$$r^4 = \lambda$$

$$r^2 = \pm\sqrt{\lambda}$$

$$r = \lambda^{1/4},\ -\lambda^{1/4},\ i\lambda^{1/4}\ -i\lambda^{1/4}$$

Introduce $\mu := \lambda^{1/4}$

$$r = \pm\mu,\ \pm i\mu$$

Hence the four linear independent solutions are

$$e^{\mu x},\ e^{-\mu x},\ \cos(\mu x),\ \sin(\mu x).$$

and the general solution reads

$$\varphi(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} + c_3 \cos(\mu x) + c_4 \sin(\mu x).$$

boundary conditions:

(i) $0 = \varphi(0) = c_1 + c_2 + c_3$

(ii) $0 = \varphi(L) = c_1 e^{\mu L} + c_2 e^{-\mu L} + c_3 \cos(\mu L) + c_4 \sin(\mu L)$

$$\varphi''(x) = c_1 \mu^2 e^{\mu x} + c_2 \mu^2 e^{-\mu x} - c_3 \mu^2 \cos(\mu x) - c_4 \mu^2 \sin(\mu x)$$

(iii) $0 = \varphi''(0) = c_1\mu^2 + c_2\mu^2 - c_3\mu^2$

(iv) $0 = \varphi''(L) = c_1\mu^2 e^{\mu L} + c_2\mu^2 e^{-\mu L} - c_3\mu^2 \cos(\mu L) - c_4\mu^2 \sin(\mu L)$

$(i) + (iii) \Rightarrow \quad C_3 = -C_3 \quad \Rightarrow \quad \boxed{C_3 = 0}$

and $\quad C_1 = -C_2$

$(ii) + (iv) \Rightarrow \quad C_4 \mu^2 \sin(\mu L) = 0$

now, if $\quad C_4 = 0 \quad$ then from $\quad (ii) \Rightarrow C_1 = -C_2 = 0.$

hence $\quad C_4 \neq 0 \quad$ and

$$\sin(\mu L) = 0$$

$$\Rightarrow \quad \mu L = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n^{1/4} = \frac{n\pi}{L}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^4$$

If $\sin(\mu L) = 0$ it still follows from (ii) that

$C_1 = -C_2 = 0.$

Eigenfunctions $\quad \varphi_n = \sin\left(\frac{n\pi x}{L}\right).$

Time equation: $\quad G_n''(t) = -\left(\frac{n\pi}{L}\right)^4 c^2 G_n(t)$

$$G_n(t) = d_1 \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right) + d_2 \sin\left(c\left(\frac{n\pi}{L}\right)^2 t\right)$$

<u>Superposition</u>

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right) + B_n \sin\left(c\left(\frac{n\pi}{L}\right)^2 t\right) \right\} \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2}{L}\frac{L^2}{c(n\pi)^2}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$