Math 209
Assignment 9 — Solutions

---

1. Evaluate $\displaystyle\iint_S \sqrt{4y+1}\,dS$ where S is the first octant part of $y = x^2$ cut out by $2x + y + z = 1$.

   <u>*Solution*</u>
   We need a parametric representation of the surface S. In vector form this is: $\vec{r} = \langle x, x^2, z\rangle$ for $(x, z) \in D$. Now determine the normal vector:

   $$\frac{\partial \vec{r}}{\partial x} = \langle 1, 2x, 0\rangle\,, \qquad \frac{\partial \vec{r}}{\partial z} = \langle 0, 0, 1\rangle\,, \qquad \vec{N} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial z} = \langle 2x, -1, 0\rangle\,.$$

   To determine the domain D, we find the curve of intersection of the surface S: $y = x^2$ and the plane: $2x + y + x = 1$:

   $$\left.\begin{matrix} y = x^2 \\ 2x + y + z = 1 \end{matrix}\right\} \implies z = 1 - 2x - x^2,$$

   then, for the part in the first octant:

   $$\left.\begin{matrix} z = 1 - 2x - x^2 \\ z = 0 \end{matrix}\right\} \implies x^2 + 2x - 1 = 0 \implies x = -1 + \sqrt{2}.$$

   Thus, the domain D is given by $D = \{(x, z) \in \mathbb{R}^2;\ 0 \leqslant x \leqslant -1 + \sqrt{2},\ 0 \leqslant z \leqslant 1 - 2x - x^2\}$. Now to evaluate the surface integral:

   $$\begin{aligned}\iint_S \sqrt{4y+1}\,dS &= \iint_D \sqrt{4x^2+1}\;|\vec{N}|\,dz\,dx \\ &= \int_0^{-1+\sqrt{2}} \int_0^{1-2x-x^2} (4x^2+1)\,dz\,dx = \frac{1}{5}(-61 + 44\sqrt{2}). \quad \blacksquare\end{aligned}$$

2. Evaluate $\displaystyle\iint_S xy\,dS$ where S is the first octant part of $z = \sqrt{x^2 + y^2}$ cut out by $x^2 + y^2 = 1$.

   <u>*Solution*</u>
   We need a parametric representation of the surface S. Since S is a surface of revolution we can use polar coordinates, so in vector form this is: $\vec{r} = \langle t\cos\theta, t\sin\theta, t\rangle$ for $(t, \theta) \in D$, where we have used t in place of r to avoid confusion with the position vector $\vec{r}$. Now determine the normal vector:

   $$\frac{\partial \vec{r}}{\partial t} = \langle \cos\theta, \sin\theta, 1\rangle\,, \qquad \frac{\partial \vec{r}}{\partial \theta} = \langle -t\sin\theta, t\cos\theta, 0\rangle\,, \qquad \vec{N} = \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial \theta} = \langle -t\cos\theta, -t\sin\theta, t\rangle\,.$$

   The domain D consists of the portion of the surface S: $z = \sqrt{x^2 + y^2}$ that lies within the first octant and within the cylinder $x^2 + y^2 = 1$. In polar coordinates (t, θ), this gives $D = \{(t, \theta) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant 1,\ 0 \leqslant \theta \leqslant \pi/2\}$. Now to evaluate the surface integral:

   $$\iint_S xy\,dS = \iint_D t^2 \sin\theta\cos\theta\;|\vec{N}|\,dt\,d\theta = \int_0^{\pi/2} \int_0^1 \sqrt{2}t^3 \sin\theta\cos\theta\,dt\,d\theta = \frac{\sqrt{2}}{8}. \quad \blacksquare$$

3. Calculate the surface area of the curved portion of a right circular cone of radius R and height h.

<u>*Solution*</u>
We need a parametric representation of the surface S. Since S is a surface of revolution we can use polar coordinates, so in vector form this is: $\vec{r} = \left\langle t\cos\theta, t\sin\theta, \frac{ht}{R}\right\rangle$ for $(t,\theta) \in D$, where we have used t in place of r to avoid confusion with the position vector $\vec{r}$. Now determine the normal vector:

$$\frac{\partial \vec{r}}{\partial t} = \left\langle \cos\theta, \sin\theta, \frac{h}{R}\right\rangle, \quad \frac{\partial \vec{r}}{\partial \theta} = \langle -t\sin\theta, t\cos\theta, 0\rangle, \quad \vec{N} = \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial \theta} = \left\langle -\frac{ht}{R}\cos\theta, -\frac{ht}{R}\sin\theta, t\right\rangle.$$

The domain D consists of the portion of the surface S: $z = (h/R)\sqrt{x^2+y^2}$ that lies within the cylinder $x^2+y^2 = R^2$. In polar coordinates (t,θ), this gives $D = \{(t,\theta) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant R,\ 0 \leqslant \theta \leqslant 2\pi\}$. Now to evaluate the surface integral:

$$\iint\limits_S dS = \iint\limits_D |\vec{N}|\,dt\,d\theta = \int_0^{2\pi}\int_0^R t\sqrt{1+(h^2/R^2)}\,dt\,d\theta = \pi R\sqrt{R^2+h^2}. \quad \blacksquare$$

4. Evaluate $\displaystyle\iint\limits_S \frac{dS}{x^2+y^2}$ where S is the part of the sphere $x^2+y^2+z^2 = 4R^2$ between the planes $z=0$ and $z=R$.

<u>*Solution*</u>
We need a parametric representation of the sphere S. The easiest way to paremeterize the sphere is to use spherical coordinates with $\rho = 2R$, so in vector form this is: $\vec{r} = 2R\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$ for $(t,\theta) \in D$. Now determine the normal vector:

$$\frac{\partial \vec{r}}{\partial \varphi} = 2R\langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\rangle, \qquad \frac{\partial \vec{r}}{\partial \theta} = 2R\langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\rangle,$$

$$\vec{N} = \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial \theta} = 4R^2\left\langle -\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi\right\rangle.$$

The domain D consists of the portion of the surface S that lies within the planes $z=0$ and $z=R$. In spherical coordinates these correspond to $\varphi = \pi/2$ and $\varphi = \pi/3$ respectively. Thus, the domain D is: $D = \{(\varphi,\theta) \in \mathbb{R}^2;\ \pi/3 \leqslant \varphi \leqslant \pi/2,\ 0 \leqslant \theta \leqslant 2\pi\}$. Now to evaluate the surface integral:

$$\iint\limits_S \frac{dS}{x^2+y^2} = \iint\limits_D \frac{|\vec{N}|}{4R^2\sin^2\varphi}\,d\varphi\,d\theta = \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \csc\varphi\,d\varphi\,d\theta = \pi\ln 3. \quad \blacksquare$$

5. Evaluate $\displaystyle\iint_S (yz^2\,\vec{i} + ye^x\,\vec{j} + x\,\vec{k}) \cdot \vec{n}\,dS$ where S is defined by $y = x^2$, $0 \leqslant y \leqslant 4$, $0 \leqslant z \leqslant 1$, and $\vec{n}$ is the unit normal to the surface S with positive y-component.

<u>Solution</u>
We need a parametric representation of the surface S. In vector form this is: $\vec{r} = \left\langle x, x^2, z \right\rangle$ for $(x,z) \in D$, where $D = \{(x,z) \in \mathbb{R}^2;\ -2 \leqslant x \leqslant 2,\ 0 \leqslant z \leqslant 1\}$. Now determine the normal vector:

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 2x, 0 \rangle\,, \qquad \frac{\partial \vec{r}}{\partial z} = \langle 0, 0, 1 \rangle\,, \qquad \vec{N} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial z} = \langle 2x, -1, 0 \rangle\,.$$

Now, to evaluate the surface integral:

$$\begin{aligned}\iint_S (yz^2\,\vec{i} + ye^x\,\vec{j} + x\,\vec{k}) \cdot \vec{n}\,dS &= \iint_D (x^2z^2\,\vec{i} + x^2e^x\,\vec{j} + x\,\vec{k}) \cdot \vec{N}\,dx\,dz \\ &= \int_0^1 \int_{-2}^2 (-2x^3z + x^2e^x)\,dx\,dz = 2e^2 - 10e^{-2}. \quad \blacksquare\end{aligned}$$

6. Evaluate $\displaystyle\iint_S (x\,\vec{i} + y\,\vec{j}) \cdot \vec{n}\,dS$ where S is the part of $z = \sqrt{x^2+y^2}$ below $z = 1$, and $\vec{n}$ is the unit normal to the surface S with negative z-component.

<u>Solution</u>
We need a parametric representation of the surface S. Since S is a surface of revolution we can use polar coordinates, so in vector form this is: $\vec{r} = \langle t\cos\theta, t\sin\theta, t \rangle$ for $(t,\theta) \in D$, where we have used t in place of r to avoid confusion with the position vector $\vec{r}$. Now determine the normal vector:

$$\frac{\partial \vec{r}}{\partial t} = \langle \cos\theta, \sin\theta, 1 \rangle\,, \qquad \frac{\partial \vec{r}}{\partial \theta} = \langle -t\sin\theta, t\cos\theta, 0 \rangle\,, \qquad \vec{N} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial t} = \langle t\cos\theta, -t\sin\theta, -t \rangle\,.$$

The domain D consists of the portion of the surface S that lies below $z = 1$, i.e. in the region $x^2+y^2 \leqslant 1$. In polar coordinates (t,θ), this gives $D = \{(t,\theta) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi\}$. Now to evaluate the surface integral:

$$\iint_S (x\,\vec{i} + y\,\vec{j}) \cdot \vec{n}\,dS = \iint_D (t\cos\theta\,\vec{i} + t\sin\theta\,\vec{j}) \cdot \vec{N}\,dt\,d\theta = \int_0^{2\pi} \int_0^1 t\,dt\,d\theta = \frac{2\pi}{3}. \quad \blacksquare$$

7. Evaluate $\displaystyle\iint_S (x^2y\vec{i} + xy\vec{j} + z\vec{k}) \cdot \vec{n}\, dS$ where S is defined by $z = 2 - x^2 - y^2$, $z \geqslant 0$, and $\vec{n}$ is the unit normal to the surface S with negative z-component.

<u>*Solution*</u>
We need a parametric representation of the surface S. Since S is a surface of revolution we can use polar coordinates, so in vector form this is: $\vec{r} = \left\langle t\cos\theta, t\sin\theta, 2 - t^2\right\rangle$ for $(t,\theta) \in D$, where we have used t in place of r to avoid confusion with the position vector $\vec{r}$. Now determine the normal vector:

$$\frac{\partial\vec{r}}{\partial t} = \langle\cos\theta, \sin\theta, -2t\rangle\,, \quad \frac{\partial\vec{r}}{\partial\theta} = \langle -t\sin\theta, t\cos\theta, 0\rangle\,, \quad \vec{N} = \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial t} = \left\langle -2t^2\cos\theta, -2t^2\sin\theta, -t\right\rangle.$$

The domain D consists of the portion of the surface S for which $z \geqslant 0$, i.e. in the region $x^2 + y^2 \leqslant 2$. In polar coordinates (t,θ), this gives $D = \{(t,\theta) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant \sqrt{2},\ 0 \leqslant \theta \leqslant 2\pi\}$. Now to evaluate the surface integral:

$$\begin{aligned}
\iint_S (x^2y\vec{i} + xy\vec{j} + z\vec{k}) \cdot \vec{n}\, dS &= \iint_D (t^3\cos^2\theta\sin\theta\,\vec{i} + t^2\cos\theta\sin\theta\,\vec{j} + (2-t^2)\vec{k}) \cdot \vec{N}\, dt\, d\theta \\
&= \int_0^{\sqrt{2}}\int_0^{2\pi} (-2t^5\cos^3\theta\sin\theta - 2t^4\cos\theta\sin^2\theta - 2t + t^3)\, d\theta\, dt \\
&= \int_0^{\sqrt{2}} \left(t^5\frac{\cos^4\theta}{2} - 2t^4\frac{\sin^3\theta}{3} + (t^3 - 2t)\theta \right)\Bigg|_0^{2\pi} dt \\
&= 2\pi\int_0^{\sqrt{2}} (t^3 - 2t)\, dt = -2\pi. \quad \blacksquare
\end{aligned}$$

8. Find the centroid of the surface S consisting of the part of $z = 2 - x^2 - y^2$ above the xy-plane.

<u>*Solution*</u>
We need a parametric representation of the surface S. Since S is a surface of revolution we can use polar coordinates, so in vector form this is: $\vec{r} = \left\langle t\cos\theta, t\sin\theta, 2 - t^2\right\rangle$ for $(t,\theta) \in D$, where we have used t in place of r to avoid confusion with the position vector $\vec{r}$. Now determine the normal vector:

$$\frac{\partial\vec{r}}{\partial t} = \langle\cos\theta, \sin\theta, -2t\rangle\,, \quad \frac{\partial\vec{r}}{\partial\theta} = \langle -t\sin\theta, t\cos\theta, 0\rangle\,, \quad \vec{N} = \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial t} = \left\langle -2t^2\cos\theta, -2t^2\sin\theta, -t\right\rangle.$$

The domain D consists of the portion of the surface S for which $z \geqslant 0$, i.e. in the region $x^2 + y^2 \leqslant 2$. In polar coordinates (t,θ), this gives $D = \{(t,\theta) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant \sqrt{2},\ 0 \leqslant \theta \leqslant 2\pi\}$. The area of S is given by:

$$A(S) = \iint_S dS = \iint_D |\vec{N}|\, dt\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} t\sqrt{1 + 4t^2}\, dt\, d\theta = \frac{13\pi}{3}.$$

By symmetry, it follows that $\bar{x} = \bar{y} = 0$. Thus, z-component of the centroid is:

$$\bar{z} = \frac{1}{A(S)}\iint_S z\, dS = \frac{1}{A(S)}\iint_D (2 - t^2)\, |\vec{N}|\, dt\, d\theta = \frac{1}{A(S)}\int_0^{2\pi}\int_0^{\sqrt{2}} t(2 - t^2)\sqrt{1 + 4t^2}\, dt\, d\theta = \frac{27\pi}{2A(S)}.$$

Thus, the centroid is: $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{111}{130})$. $\blacksquare$

9. Find the moment of inertia about the z-axis of the surface S consisting of the part of $z = 2 - x^2 - y^2$ above the xy-plane.

<u>*Solution*</u>

We need a parametric representation of the surface S. Since S is a surface of revolution we can use polar coordinates, so in vector form this is: $\vec{r} = \left\langle t\cos\theta, t\sin\theta, 2 - t^2 \right\rangle$ for $(t, \theta) \in D$, where we have used t in place of r to avoid confusion with the position vector $\vec{r}$. Now determine the normal vector:

$$\frac{\partial \vec{r}}{\partial t} = \langle \cos\theta, \sin\theta, -2t \rangle\,, \quad \frac{\partial \vec{r}}{\partial \theta} = \langle -t\sin\theta, t\cos\theta, 0 \rangle\,, \quad \vec{N} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial t} = \left\langle -2t^2\cos\theta, -2t^2\sin\theta, -t \right\rangle.$$

The domain D consists of the portion of the surface S for which $z \geqslant 0$, i.e. in the region $x^2 + y^2 \leqslant 2$. In polar coordinates (t, θ), this gives $D = \{(t,\theta) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant \sqrt{2},\ 0 \leqslant \theta \leqslant 2\pi\}$. The moment of inertia about the z-axis is:

$$I_z = \frac{1}{A(S)} \iint_S (x^2 + y^2)\, dS = \iint_D t^2\, |\vec{N}|\, dt\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} t^3 \sqrt{1 + 4t^2}\, dt\, d\theta = \frac{149\pi}{130}. \quad \blacksquare$$

10. A circular tube $S:\ x^2 + z^2 = 1$, $0 \leqslant y \leqslant 2$ is a model for a part of an artery. Blood flows through the artery and the force per unit area at any point on the arterial wall is given by

$$\vec{F} = e^{-y}\vec{n} + \frac{1}{y^2+1}\vec{j}\,,$$

where $\vec{n}$ is the unit outer normal to the arterial wall. Blood diffuses through the wall in such a way that if dS is a small area on S, the amount of diffusion through dS in one second is $\vec{F} \cdot \vec{n}\, dS$. Find the total amount of blood leaving the entire wall per second.

<u>*Solution*</u>

We need a parametric representation of the surface S. Since S is a cylinder with axis parallel to the y-axis, the most prudent way to parameterize the surface is by using polar coordinate θ in the xz-plane and y as the parameters, so in vector form this is: $\vec{r} = \langle \cos\theta, y, \sin\theta \rangle$ for $(y, \theta) \in D$, where $D = \{(y,\theta) \in \mathbb{R}^2;\ 0 \leqslant y \leqslant 2,\ 0 \leqslant \theta \leqslant 2\pi\}$. Now determine the normal vector:

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, 0 \rangle\,, \quad \frac{\partial \vec{r}}{\partial \theta} = \langle -\sin\theta, 0, \cos\theta \rangle\,, \quad \vec{N} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial t} = \langle \cos\theta, 0, \sin\theta \rangle\,.$$

The unit normal $\vec{n}$ and surface element dS are given by:

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \langle \cos\theta, 0, \sin\theta \rangle\,, \qquad dS = |\vec{N}|\, dy\, d\theta = dy\, d\theta.$$

The total amount of blood leaving the entire wall per second is:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \left(e^{-y}\vec{n} + \frac{1}{y^2+1}\vec{j} \right) \cdot \vec{n}\, dy\, d\theta = \int_0^{2\pi} \int_0^2 e^{-y}\, dy\, d\theta = 2\pi(1 - e^{-2}). \quad \blacksquare$$