Math 209

Solutions to Assignment 7

Due: 12 Noon on Thursday, November 10, 2005

1. Find the gradient vector field of the following functions:

(a) $f(x, y) = \ln(x + 2y)$;

(b) $f(x, y, z) = x \cos(y/z)$.

Solution. (a)

$$f_x = \frac{1}{x+2y}, \qquad f_y = \frac{2}{x+2y}.$$

Thus, the gradient vector field is

$$\nabla f = \Big(\frac{1}{x+2y}, \frac{2}{x+2y}\Big) = \frac{1}{x+2y}\,\mathbf{i} + \frac{2}{x+2y}\,\mathbf{j}.$$

(b)

$$f_x = \cos(y/z)), \quad f_y = -\frac{x}{z}\sin(y/z), \qquad f_z = \frac{xy}{z^2}\sin(y/z).$$

Thus, the gradient vector field is

$$\begin{aligned}\nabla f &= \Big(\cos(y/z)), -\frac{x}{z}\sin(y/z), \frac{xy}{z^2}\sin(y/z)\Big)\\ &= \cos(y/z))\,\mathbf{i} - \frac{x}{z}\sin(y/z)\,\mathbf{j} + \frac{xy}{z^2}\sin(y/z)\,\mathbf{k}.\end{aligned}$$

2. Suppose $f(x, y) = x^2 - y^2$. Find $\displaystyle\int_C f\,ds$ where

(a) C is formed from the edges of a triangle with vertices at $(0, 0)$, $(2, 1)$ and $(1, 2)$.

(b) C is a circle of radius 2 centered at the origin.

Solution. (a) For convenience, we denote by C_1, C_2 and C_3 the line segments from $(0, 0)$ to $(2, 1)$, from $(2, 1)$ to $(1, 2)$ and from $(1, 2)$ to $(0, 0)$, respectively. Then $C = C_1+C_2+C_3$ and the parametric equations of C_1, C_2 and C_3 as given as follows:

$$C_1: \begin{cases} x = 0\cdot(1-t) + 2t = 2t \\ y = 0\cdot(1-t) + t = t, \end{cases} \qquad 0 \le t \le 1,$$

$$C_2: \begin{cases} x = 2\cdot(1-t) + t = 2 - t \\ y = 1(1-t) + 2t = t + 1, \end{cases} \qquad 0 \le t \le 1,$$

$$C_3: \begin{cases} x = 1\cdot(1-t) + 0\cdot t = 1 - t \\ y = 2(1-t) + 0\cdot t = 2 - 2t, \end{cases} \qquad 0 \le t \le 1,$$

Thus,

$$\begin{aligned}\int_C f\,ds &= \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds\\ &= \int_0^1 (4t^2 - t^2)\sqrt{5}\,dt + \int_0^1 \Big((2-t)^2 - (t+1)^2\Big)\sqrt{2}\,dt + \int_0^1 \Big((1-t)^2 - 4(1-t)^2\Big)\sqrt{5}\,dt\\ &= 3\sqrt{5}\int_0^1 t^2\,dt + \sqrt{2}\int_0^1 (3 - 6t)\,dt + \sqrt{5}\int_0^1 (-3 + 6t - 3t^2)\,dt\\ &= 0.\end{aligned}$$

(b) The paramentric equation of C is

$$\begin{cases} x &= 2\cos t \\ y &= 2\sin t, \end{cases} \quad 0 \le t \le 2\pi.$$

Thus

$$\begin{aligned} \int_C f\,ds &= \int_0^{2\pi} \left(4\cos^2 t - 4\sin^2 t\right)\sqrt{4\sin^2 t + 4\cos^2 t}\,dt \\ &= 8\int_0^{2\pi} \cos(2t) = 4\sin(2t)\Big|_0^{2\pi} = 0. \end{aligned}$$

3. Evaluate $\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$, where C consists of line segments from $(1,0,1)$ to $(2,3,1)$ and from $(2,3,1)$ to $(2,5,2)$.

Solution. We denote by C_1 and C_2 the line segments from $(1,0,1)$ to $(2,3,1)$ and from $(2,3,1)$ to $(2,5,2)$, respectively. Then

$$\begin{aligned} C_1:\ (x(t),y(t),z(t)) &= (1,0,1)(1-t) + t(2,3,1) = (1+t, 3t, 1), \quad 0 \le t \le 1, \\ C_2:\ (x(t),y(t),z(t)) &= (2,3,1)(1-t) + t(2,5,2) = (2, 3+2t, 1+t), \quad 0 \le t \le 1. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{C_1} (x+yz)\,dx + 2x\,dy + xyz\,dz &= \int_0^1 \Big((1+t+3t) + 2(1+t)\cdot 3 + 3t(1+t)\cdot 0\Big)\,dt \\ &= \int_0^1 (7+10t)\,dt = 12, \\ \int_{C_2} (x+yz)\,dx + 2x\,dy + xyz\,dz &= \int_0^1 \Big((2+(3+2t)(1+t))\cdot 0 + 8 + 2(3+2t)(1+t)\Big)\,dt \\ &= \int_0^1 (14+10t+4t^2)\,dt = \frac{61}{3}. \end{aligned}$$

Therefore

$$\int_C = \int_{C_1} + \int_{C_2} = 12 + \frac{61}{3} = \frac{97}{3}.$$

4. The formula for a cycloid is given parametrically by $(t-\sin(t), 1-\cos(t))$. Find the length of the curve over one cycle $0 \le t \le 2\pi$.

Solution. The length is

$$\begin{aligned} \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\,dt \\ &= \int_0^{2\pi} \sqrt{2-2\cos t}\,dt = 2\int_0^{2\pi} \sin\frac{t}{2}\,dt = 8. \end{aligned}$$

5. Determine whether or not $\mathbf{F}$ is a conservative vector field, if it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\quad \mathbf{F}(x,y) = (2x\cos y - y\cos x)\mathbf{i} + (-x^2\sin y - \sin x)\mathbf{j}$.

   (b) $\quad \mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$.

Solution. (a) Let

$$P(x,y) = 2x\cos y - y\cos x, \quad Q(x,y) = -x^2\sin y - \sin x.$$

Since

$$P_y = -2x\sin y - \cos x = Q_x$$

throughout the open, simply connected domain $\mathbb{R}^2$, it follows that $\mathbf{F}$ is conservative.
Now assume $\nabla f = \mathbf{F}$. Then

$$f_x = P = 2x\cos y - y\cos x,$$

and hence

$$f = \int P(x,y)\,dx + g(y) = x^2\cos y - y\sin x + g(y).$$

On the the hand, since $f_y = Q$,

$$\frac{\partial}{\partial y}\Big(x^2\cos y - y\sin x + g(y)\Big) = -x^2\sin y - \sin x,$$

that is

$$-x^2\sin y - \sin x + g'(y) = -x^2\sin y - \sin x.$$

Thus, $g'(y) = 0$ and $g(y) = K$, where K is a constant. Therefore

$$f(x,y) = x^2\cos y - y\sin x + K.$$

(b) Let

$$P(x,y) = ye^x + \sin y, \quad Q(x,y) = e^x + x\cos y.$$

Then

$$P_y = e^x + \cos y = Q_x$$

throughout $\mathbb{R}^2$. Thus, $\mathbf{F}$ is conservative.
Now assume $\nabla f = \mathbf{F}$. Then

$$f(x,y) = \int P(x,y)\,dx + g(y) = ye^x + x\sin y + g(y).$$

Since $f_y = Q$,

$$\frac{\partial}{\partial y}\Big(ye^x + x\sin y + g(y)\Big) = e^x + x\cos y.$$

It follows that $g'(y) = 0$ and hence $g(y) = K$. Therefore

$$f(x,y) = ye^x + x\sin y + K.$$

6. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ along the given curve C:

(*a*) $\mathbf{F}(x,y) = \Big(\dfrac{y^2}{1+x^2}\Big)\mathbf{i} + \Big(2y\arctan x\Big)\mathbf{j}$, $\quad C:\ \ \mathbf{r}(t) = (t^2)\ \mathbf{i} + (2t)\ \mathbf{j}$, $\quad 0 \le t \le 1$.

(*b*) $\mathbf{F}(x,y,z) = (y^2\cos z)\mathbf{i} + (2xy\cos z)\mathbf{j} - (xy^2\sin z)\mathbf{k}$, $\quad C:\ \ \mathbf{r}(t) = (t^2)\ \mathbf{i} + (\sin t)\ \mathbf{j} + t\ \mathbf{k}$, $0 \le t \le \pi$.

Solution. (a) Let

$$P(x,y) = \frac{y^2}{x^2+1}, \quad Q(x,y) = 2y\arctan x.$$

Then it's easy to verify that

$$\nabla f = \mathbf{F} = (P, Q),$$

where

$$f(x, y) = y^2 \arctan x.$$

It follows by the fundamental theorem that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1, 2) - f(0, 0) = \pi.$$

(b) Let

$$f(x, y, z) = xy^2 \cos z.$$

Then it's easy to verify that

$$\nabla f = (y^2 \cos z, 2xy \cos z, -xy^2 \sin z) = \mathbf{F}.$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(\pi^2, 0, \pi) - f(0, 0, 0) = 0.$$

7. Show that the line integral is independent of path and evaluate the integral:

$$\int_C (1 - ye^{-x})\, dx + e^{-x}\, dy,$$

where C is any path from $(0, 1)$ to $(1, 2)$.

Solution. Let

$$\mathbf{F} = (1 - ye^{-x}, e^{-x}) \equiv (P, Q).$$

Suppose the equation of C is given by

$$\mathbf{r} = \mathbf{r}(t).$$

Then

$$\int_C (1 - ye^{-x})\, dx + e^{-x}\, dy = \int_C P\, dx + Q\, dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Let

$$f(x, y) = x + ye^{-x}.$$

then it is easy to verify that

$$\nabla f = \mathbf{F}.$$

Therefore $\mathbf{F}$ is conservative and the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. Moreover, by the fundamental theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1, 2) - f(0, 1) = 2e^{-1}.$$

8. Find the work done by the force field $\mathbf{F}(x, y) = (y^2/x^2)\ \mathbf{i} - (2y/x)\ \mathbf{j}$ in moving an object from $P(1, 1)$ to $Q(4, -2)$.

Solution. Let

$$f(x, y) = -\frac{y^2}{x}.$$

Then

$$\nabla f = (\frac{y^2}{x^2}, -\frac{2y}{x}) = \mathbf{F}.$$

Therefore the work done by the force field $\mathbf{F}(x,y) = (y^2/x^2)\ \mathbf{i} - (2y/x)\ \mathbf{j}$ in moving an object from $P(1,1)$ to $Q(4,-2)$ is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(4,-2) - f(1,1) = -1 - (-1) = 0.$$

9. Show that if the vector field $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Solution. Suppose

$$\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}$$

is conservative. Then there exists a function $f(x,y,z)$ such that

$$f_x = P, \quad f_y = Q, \quad f_z = R.$$

Therefore

$$\begin{aligned} \frac{\partial P}{\partial y} &= f_{xy}, & \frac{\partial Q}{\partial x} &= f_{yx}, \\ \frac{\partial P}{\partial z} &= f_{xz}, & \frac{\partial R}{\partial x} &= f_{zx}, \\ \frac{\partial Q}{\partial z} &= f_{yz}, & \frac{\partial R}{\partial y} &= f_{zy}. \end{aligned}$$

It then follows by Clairaut's theorem that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

10. Let $\mathbf{F}(x,y) = \dfrac{-y\ \mathbf{i} + x\ \mathbf{j}}{x^2+y^2}$.

(a) Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) Show that $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is not independent of path. [Hint: Consider the upper and lower halves of the circle $x^2+y^2=1$ from $(1,0)$ to $(-1,0)$]

Proof. (a) Since

$$P(x,y) = \frac{-y}{x^2+y^2}, \quad Q(x,y) = \frac{x}{x^2+y^2},$$

$$\begin{aligned} \frac{\partial P}{\partial y} &= -\frac{1}{x^2+y^2} + \frac{y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \\ \frac{\partial Q}{\partial x} &= \frac{1}{x^2+y^2} - \frac{x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}. \end{aligned}$$

Thus

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

(b) We denote by C_1 and C_2 the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$, respectively. Then

$$\begin{aligned} C_1 :\ & \mathbf{r}(t) = (\cos t, \sin t), \quad 0 \le t \le \pi, \\ C_2 :\ & \mathbf{r}(t) = (\cos t, -\sin t),\ 0 \le t \le \pi. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy \\ &= \int_0^\pi \Big((-\sin t)(-\sin t) + (\cos t \cdot \cos t) \Big)\, dt = \pi, \\ \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_2} \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy \\ &= \int_0^\pi \Big((\sin t)(-\sin t) + (\cos t)(-\cos t) \Big)\, dt = -\pi. \end{aligned}$$

Thus $\displaystyle\int_{C_1} \ne \int_{C_2}$ and $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is dependant on path. (Note: This happens because the domain of definition of $\mathbf{F}$ is $\mathbb{R}^2 \setminus \{(0, 0)\}$, which is not simply connected.)