Math 209

Assignment 2 — Solutions

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1. Let $R = \ln(u^2 + v^2 + w^2)$, $u = x + 2y$, $v = 2x - y$, and $w = 2xy$. Use the Chain Rule to find $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$ when $x = y = 1$.

   Solution:

   The Chain Rule gives

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{2u}{u^2+v^2+w^2} \times 1 + \frac{2v}{u^2+v^2+w^2} \times 2 + \frac{2w}{u^2+v^2+w^2} \times (2y).$$

   When $x = y = 1$, we have $u = 3$, $v = 1$, and $w = 2$, so

$$\frac{\partial R}{\partial x} = \frac{6}{14} \times 1 + \frac{2}{14} \times 2 + \frac{4}{14} \times 2 = \frac{18}{14} = \frac{9}{7}.$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{2u}{u^2+v^2+w^2} \times 2 + \frac{2v}{u^2+v^2+w^2} \times (-1) + \frac{2w}{u^2+v^2+w^2} \times (2x).$$

   When $x = y = 1$, we have $u = 3$, $v = 1$, and $w = 2$, so

$$\frac{\partial R}{\partial x} = \frac{6}{14} \times 2 + \frac{2}{14} \times (-1) + \frac{4}{14} \times 2 = \frac{18}{14} = \frac{9}{7}.$$

2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xyz = \sin(x + y + z)$.

   Solution:

   Let $F(x, y, z) = xyz - \sin(x + y + z) = 0$. Then, we have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{yz - \cos(x+y+z)}{xy - \cos(x+y+z)}\,,$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{xz - \cos(x+y+z)}{xy - \cos(x+y+z)}\,.$$

3. Let f and g be two differentiable real valued functions. Show that any function of the form $z = f(x+at) + g(x-at)$ is a solution of the wave equation $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$.

Solution:

Let $u = x + at$ and $v = x - at$. Then $z = f(u) + g(v)$ and the Chain Rule gives

$$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial u}{\partial x} = \frac{df}{du} + \frac{dg}{dv}.$$

Thus

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{df}{du} + \frac{dg}{dv}\right) = \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}. \qquad (1)$$

Similarly

$$\frac{\partial z}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\frac{\partial v}{\partial t} = a\frac{df}{du} + a\frac{dg}{dv}.$$

Thus

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial x}\left(a\frac{df}{du} + a\frac{dg}{dv}\right) = a^2\frac{d^2 f}{du^2} + a^2\frac{d^2 g}{dv^2} = a^2\left(\frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}\right). \qquad (2)$$

From Equations (1) and (2) we get

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}.$$

4. A function f is called **homogeneous of degree n** if it is satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer. Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

[**Hint:** Use the Chain Rule to differentiate $f(tx, ty)$ with respect t.]

Solution:

Let $u = tx$ and $v = ty$. Then

$$\frac{d}{dt}\left(f(u, v)\right) = nt^{n-1}f(x, y).$$

The Chain Rule gives

$$\frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} = nt^{n-1}f(x, y).$$

Therefore

$$x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} = nt^{n-1}f(x, y). \qquad (3)$$

Setting $t = 1$ in the Equation (3):

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

5. Find the directional derivative of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(1, 2, -2)$ in the direction of vector $\mathbf{v} = \langle -6, 6, -3 \rangle$.

Solution:

We first compute the gradient vector at $(1, 2, -2)$.

$$\nabla f(x, y, z) = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$$

$$\nabla f(1, 2, -2) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \right\rangle.$$

Note that $\mathbf{v}$ is not unit vector, but since $|\mathbf{v}| = 9$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{-2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle.$$

Therefore

$$D_{\mathbf{u}} f(1, 2, -2) = \nabla f(1, 2, -2) \cdot \mathbf{u} = \frac{2}{3}.$$

6. The temperature at a point (x, y, z) on the surface of a metal is $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$ where T is measured in degree Celsius and x, y, z in meters.

   (a) In which direction does the temperature increase fastest at the point $P(2, -1, 2)$?

   (b) What is the maximum rate of change at $P(2, -1, 2)$?

Solution:

We first compute the gradient vector:

$$\nabla T(x, y, z) = \langle T_x, T_y, T_z \rangle = -e^{-x^2-3y^2-9z^2} \langle 400x, 1200y, 3600z \rangle$$

$$\nabla T(2, -1, 2) = -400e^{-43} \langle 2, -3, 18 \rangle .$$

The temperature increases in the direction of the gradient vector

$$\nabla T(2, -1, 2) = -400e^{-43} \langle 2, -3, 18 \rangle .$$

The maximum rate of change is

$$\left| -400e^{-25} \langle 2, -3, 18 \rangle \right| = 400e^{-43}\sqrt{337} .$$

7. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - 2y + 3z = 1$.

Solution:

Let $f(x,y,z) = x^2 + 2y^2 + 3z^2$. The normal vector of the plane $3x - 2y + 3z = 1$ is $\langle 3,-2,3\rangle$. The normal vector for tangent plane at the point (x_0, y_0, z_0) on the ellipsoid is $\nabla f(x_0, y_0, z_0) = \langle 2x_0, 4y_0, 6z_0\rangle$. Since the tangent plane is parallel to the given plane, $\nabla f(x_0, y_0, z_0) = \langle 2x_0, 4y_0, 6z_0\rangle = c\langle 3,-2,3\rangle$ or $\langle x_0, 2y_0, 3z_0\rangle = k\langle 3,-2,3\rangle$. Thus $x_0 = 3k$, $y_0 = -k$ and $z_0 = k$. But $x_0^2 + 2y_0^2 + 3z_0^2 = 1$ or $(9+2+3)k^2 = 1$,so $k = \pm\frac{\sqrt{14}}{14}$ and there are two such point $(\pm\frac{\sqrt{14}}{14}, \pm\frac{\sqrt{14}}{14}, \pm\frac{\sqrt{14}}{14})$.

8. Find the local maximum and minimum values and saddle point(s) of the function

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2.$$

Solution:

The first order partial derivatives are

$$f_x = 6xy - 6x, \qquad f_y = 3x^2 + 3y^2 - 6y\,.$$

So to find the critical points we need to solve the equations $f_x = 0$ and $f_y = 0$. $f_x = 0$ implies $x = 0$ or $y = 1$ and when $x = 0$, $f_y = 0$ implies $y = 0$ or $y = 2$; when $y = 1$, $f_y = 0$ implies $x^2 = 1$ or $x = \pm 1$. Thus the critical points are $(0,0)$, $(0,2)$, $(\pm 1, 1)$.

Now $f_{xx} = 6y - 6$, $f_{yy} = 6y - 6$ and $f_{xy} = 6x$. So $D = f_{xx}f_{yy} - f_{xy}^2 = (6y-6)^2 - 36x^2$.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0,0)$	2	-6	36	local maximum
$(0,2)$	-2	6	36	local minimum
$(1,1)$	0	0	-36	saddle point
$(-1,1)$	0	0	-36	saddle point

9. Find the points on surface $x^2y^2z = 1$ that are closest to the origin.

Solution:

The distance from any point (x,y,z) to the origin is

$$d = \sqrt{x^2 + y^2 + z^2}$$

but if (x,y,z) lies on the surface $x^2y^2z = 1$, then $z = \frac{1}{x^2y^2}$ and so we have

$$d = \sqrt{x^2 + y^2 + x^{-4}y^{-4}}\,.$$

We can minimize d by minimizing the simpler expression

$$d^2 = x^2 + y^2 + x^{-4}y^{-4} = f(x,y)\,.$$

$f_x = 2x - \frac{4}{x^5y^4}$, $f_y = 2y - \frac{4}{x^4y^5}$, so the critical points occur when $2x = \frac{4}{x^5y^4}$ and $2y = \frac{4}{x^4y^5}$ or $x^6y^4 = x^4y^6$ so, $x^2 = y^2$ and $x^{10} = 2 \Rightarrow x = \pm 2^{\frac{1}{10}}$, $y = \pm 2^{\frac{1}{10}}$. The four critical points $(\pm 2^{\frac{1}{10}}, \pm 2^{\frac{1}{10}})$. Thus the points on the surface closes to origin are $(\pm 2^{\frac{1}{10}}, \pm 2^{\frac{1}{10}})$. There is no maximum since the surface is infinite in extent.

10. Find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region

$$D = \{(x, y) |\ x^2 + y^2 \leq 16\}.$$

Solution:

We first need to find the critical points. These occur when

$$f_x = 4x - 4 = 0, \qquad f_y = 6y = 0$$

so the only critical point of f is $(1, 0)$ and it lies in the region $x^2 + y^2 \leq 16$.

On the circle $x^2 + y^2 = 16$, we have $y^2 = 16 - x^2$ and

$$g(x) = f(x, \sqrt{16 - x^2}) = 2x^2 + 3(16 - x^2) - 4x - 5 = -x^2 - 4x + 43\,.$$

$$g^{'}(x) = 0 \Rightarrow -2x - 4 = 0 \Rightarrow x = -2$$

$$y^2 = 16 - x^2 = 16 - 4 = 12 \Rightarrow y = \pm 2\sqrt{3}\,.$$

Now $f(1, 0) = -7$ and $f(-2, \pm 2\sqrt{3}) = 47$. Thus the maximum value of $f(x, y)$ on the disc $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1, 0) = -7$.