Math 209
Midterm — Solutions

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1. Either compute the following limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}.$$

Solution
We have (by passing to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} &= \lim_{x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2y^2+1}+1} \\ &= \lim_{x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}\cdot\frac{1}{2} \\ &= \frac{1}{2}\lim_{r\to0}\frac{r^4\cos^2\theta\sin^2\theta}{r^2} = \frac{1}{2}\lim_{r\to0} r^2\cos^2\theta\sin^2\theta.\end{aligned}$$

By applying the squeeze property

$$0 \le r^2\cos^2\theta\sin^2\theta \le r^2 \longrightarrow 0 \quad \text{as} \quad r\to0,$$

we obtain

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = \frac{1}{2}\cdot 0 = 0. \quad \blacksquare$$

2. (a) Let $u(x,y,z) = e^{ax+4y}\cos(5z)$. Find all values of the constant "a" which make the following true:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(b) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at the point $P : (1, \ln 2, \ln 3)$ if $\quad xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0.$

Solution

(a) Taking derivatives we have:

$$\begin{aligned} \frac{\partial u}{\partial x} &= ae^{ax+4y}\cos(5z) = au, & \frac{\partial^2 u}{\partial x^2} &= a\frac{\partial u}{\partial x} = a^2u, \\ \frac{\partial u}{\partial y} &= 4e^{ax+4y}\cos(5z) = 4u, & \frac{\partial^2 u}{\partial y^2} &= 4\frac{\partial u}{\partial y} = 16u, \\ \frac{\partial u}{\partial z} &= -5e^{ax+4y}\sin(5z), & \frac{\partial^2 u}{\partial z^2} &= -5^2e^{ax+4y}\cos(5z) = -25u. \end{aligned}$$

Therfore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{when} \quad (a^2+16-25)u = 0 \quad \text{i.e. when} \quad a = \pm3.$$

(b) Let $F(x,y,z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2$. Taking derivatives we have:

$$\begin{aligned} F_x(x,y,z) &= e^y + \frac{2}{x}, & F_x(1,\ln 2,\ln 3) &= 4,\\ F_y(x,y,z) &= xe^y + e^z, & F_y(1,\ln 2,\ln 3) &= 5,\\ F_z(x,y,z) &= ye^z, & F_z(1,\ln 2,\ln 3) &= 3\ln 2. \end{aligned}$$

Therefore

$$\left.\frac{\partial z}{\partial x}\right|_P = -\frac{F_x(1,\ln 2,\ln 3)}{F_z(1,\ln 2,\ln 3)} = -\frac{4}{3\ln 2}, \qquad \left.\frac{\partial z}{\partial y}\right|_P = -\frac{F_y(1,\ln 2,\ln 3)}{F_z(1,\ln 2,\ln 3)} = -\frac{5}{3\ln 2}. \quad \blacksquare$$

3. (a) Find the equation of the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1,-1,1)$.
   (b) Find the differential du for the function $u = xy + xz + yz$.

<u>*Solution*</u>

(a) Let

$$F(x,y,z) = x^2 + 2y^2 + 3z^2.$$

Then

$$\nabla F = (F_x, F_y, F_z) = (2x, 4y, 6z),$$

which implies $\nabla F(1,-1,1) = (2,-4,6)$.
Thus, the equation of the tangent plane to the surface $F(x,y,z) = 6$ at the point $(1,-1,1)$ is

$$(2,-4,6)\cdot(x-1, y+1, z-1) = 0,$$

that is

$$2(x-1) - 4(y+1) + 6(z-1) = 0.$$

Simplifying this last equation gives

$$x - 2y + 3z = 6.$$

(b) Note that

$$u_x = y + z,\ u_y = x + z,\ u_z = x + y.$$

Thus

$$\begin{aligned} du &= u_x dx + u_y dy + u_z dz\\ &= (y+z)dx + (x+z)dy + (x+y)dz. \quad \blacksquare \end{aligned}$$

4. The Temperature at a point (x,y,z) is given by

$$T(x,y,z) = 100e^{-x^2-xy-z^3+2}$$

   (a) Find the rate of change of the temperature at the point $(1,0,1)$ in direction of $v = \langle 1,1,1\rangle$.
   (b) In which direction does the temperature increase fastest at P?
   (c) Find the maximum rate of change of the temperature at P.

<u>*Solution*</u>

(a)

$$\begin{aligned}
\nabla T(x,y,z) &= (-2x-y,-x,-3z^2)100e^{-x^2-xy-z^3+2} \\
\nabla T(1,0,1) &= 100(-2,-1,-3) \\
\|v\| &= \frac{1}{\sqrt{3}} \qquad u := (\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}) \\
D_uT(1,0,1) &= \nabla T(1,0,1)\cdot u \\
&= \frac{100}{\sqrt{3}}(-2-1-3) = -\frac{600}{\sqrt{3}} = -200\sqrt{3}.
\end{aligned}$$

(b) The temperature increases fastest in P in direction

$$\nabla T(1,0,1) = (-200,-100,-300).$$

Also correct are any positive scalar multiple of the above vector, e.g. $(-2,-1,-3)$ or $(-2/\sqrt{14},-1/\sqrt{14},-3/\sqrt{14})$ etc.

(c)

$$\|\nabla T(1,0,1)\| = 100\sqrt{4+1+9} = 100\sqrt{14}. \quad \blacksquare$$

5. (a) Find the local maximum and minimum values and the saddle points of the function $f(x,y) = (x^2+y^2)e^{y^2-x^2}$.
   (b) Find the extreme values of the function $f(x,y) = e^{-xy}$ on the region described by the inequality $x^2+4y^2 \le 1$. Use Lagrange multipliers to treat the boundary case.

<u>*Solution*</u>

(a) First, to find the critical/singular points we need to compute the first-order partial derivatives:

$$f_x = 2xe^{y^2-x^2}(1-x^2-y^2), \qquad f_y = 2ye^{y^2-x^2}(1+x^2+y^2).$$

Clearly $f_y = 0$ if ad only if $y=0$. Using this into $f_x = 0$ we get $2xe^{-x^2}(1-x^2)=0$, which gives $x=0$ or $x=\pm1$. Thus $\boxed{\text{the critical points are } (0,0),\ (1,0) \text{ and } (-1,0)}$. To classify these points we use the second derivative test. Note that

$$\begin{aligned}
f_{xx} &= 2e^{y^2-x^2}\left((1-x^2-y^2)(1-2x^2)-2x^2\right), \\
f_{xy} &= f_{yx} = -4xye^{y^2-x^2}(x^2+y^2), \\
f_{yy} &= 2e^{y^2-x^2}\left((1+x^2+y^2)(1+2y^2)+2y^2\right).
\end{aligned}$$

The Hessian matrix at (0,0) is $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$ which is positive definite, therefore (0,0) gives a local minimum (turns out to be the global minimum), and its value is $f(0,0)=0$.

The Hessian matrix at $(\pm1,0)$ is $\begin{pmatrix}-4e^{-1} & 0\\ 0 & 4e^{-1}\end{pmatrix}$ which is indefinite, therefore $(\pm1,0)$ are saddle points.

(b) Since the region is closed and bounded, the function attains its max. and min. there. All we have to do is find the critical points, singular points, etc. and evaluate the function at those points. First we work **in the interior** ($x^2+4y^2<1$). Here we find the critical points. We see that $\nabla f(x,y) = (-ye^{-xy},-xe^{-xy})$ equals $(0,0)$ if and only if $x=0$ and $y=0$. Thus $(0,0)$ is the only critical point in the interior. At this point we have $\boxed{f(0,0)=1}$. There are no singular points.

Second, we work **on the boundary** ($x^2 + 4y^2 = 1$). Here we use Lagrange multipliers with $f(x, y) = e^{-xy}$ being the objective function, and $g(x, y) := x^2 + 4y^2 = 1$ the constraint. From $\nabla f = \lambda \nabla g$ we get the equations

$$-ye^{-xy} = 2\lambda x \tag{A}$$

$$-xe^{-xy} = 8\lambda y. \tag{B}$$

Combining these two equations we get $x^2 = 4y^2$, and using this into the constraint $x^2 + 4y^2 = 1$ we obtain $x = \pm\frac{1}{\sqrt{2}}$ and therefore $y = \pm\frac{1}{2\sqrt{2}}$. We evaluate the function at these points:

$$f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{\frac{1}{4}}, \qquad f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-\frac{1}{4}}.$$

Since $e^{-\frac{1}{4}} < 1 < e^{\frac{1}{4}}$, the maximum of f on the region is $e^{\frac{1}{4}}$, and the minimum is $e^{-\frac{1}{4}}$. ∎

6. Find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the triangular region in the xy-plane bounded by the lines $y = x$, $x = 0$, and $x + y = 2$.

Solution

$$V = \iint_D (x^2 + y^2)dxdy\,,$$

where $D = \{(x, y) \mid x \leq y \leq 2 - x \quad \text{and} \quad 0 \leq x \leq 1\}$.

$$V = \iint\limits_D (x^2 + y^2)dxdy = \int_0^1 \int_x^{2-x} (x^2 + y^2)dxdy = \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_x^{2-x} dx$$

$$= \int_0^1 \left(x^2(2 - x) + \frac{(2 - x)^3}{3} - x^3 - \frac{x^3}{3}\right) dx = \int_0^1 \left(-\frac{7x^3}{3} + 2x^2 + \frac{(2 - x)^3}{3}\right) dx$$

$$= \left(-\frac{7x^4}{12} + \frac{2x^3}{3} - \frac{(2 - x)^4}{12}\right)_0^1 = -\frac{7}{12} + \frac{2}{3} - \frac{1}{12} + \frac{16}{12} = \frac{16}{12} = \frac{4}{3}\,. \quad ∎$$