Math 209

Assignment 7

Due: 12 Noon on Thursday, November 10, 2005

1. Find the gradient vector field of the following functions:
   (a) $f(x, y) = \ln(x + 2y)$;
   (b) $f(x, y, z) = x\cos(y/z)$.

2. Suppose $f(x, y) = x^2 - y^2$. Find $\int_C f\, ds$ where
   (a) C is formed from the edges of a triangle with vertices at $(0, 0)$, $(2, 1)$ and $(1, 2)$.
   (b) C is a circle of radius 2 centered at the origin.

3. Evaluate $\int_C (x + yz)\, dx + 2x\, dy + xyz\, dz$, where C consists of line segments from $(1, 0, 1)$ to $(2, 3, 1)$ and from $(2, 3, 1)$ to $(2, 5, 2)$.

4. The formula for a cycloid is given parametrically by $(t - \sin(t), 1 - \cos(t))$. Find the length of the curve over one cycle $0 \le t \le 2\pi$.

5. Determine whether or not $\mathbf{F}$ is a conservative vector field, if it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y) = (2x\cos y - y\cos x)\mathbf{i} + (-x^2 \sin y - \sin x)\mathbf{j}$.

   (b) $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$.

6. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C:

   (a) $\mathbf{F}(x, y) = \Big(\dfrac{y^2}{1 + x^2}\Big)\mathbf{i} + \Big(2y \arctan x\Big)\mathbf{j}$, $C:$ $\mathbf{r}(t) = (t^2)\ \mathbf{i} + (2t)\ \mathbf{j}$, $0 \le t \le 1$.

   (b) $\mathbf{F}(x, y, z) = (y^2 \cos z)\mathbf{i} + (2xy\cos z)\mathbf{j} - (xy^2 \sin z)\mathbf{k}$, $C:$ $\mathbf{r}(t) = (t^2)\ \mathbf{i} + (\sin t)\ \mathbf{j} + t\ \mathbf{k}$, $0 \le t \le \pi$.

7. Show that the line integral is independent of path and evaluate the integral:

$$\int_C (1 - ye^{-x})\, dx + e^{-x}\, dy,$$

   where C is any path from $(0, 1)$ to $(1, 2)$.

8. Find the work done by the force field $\mathbf{F}(x, y) = (y^2/x^2)\ \mathbf{i} - (2y/x)\ \mathbf{j}$ in moving an object from $P(1, 1)$ to $Q(4, -2)$.

9. Show that if the vector field $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

10. Let $\mathbf{F}(x, y) = \dfrac{-y\ \mathbf{i} + x\ \mathbf{j}}{x^2 + y^2}$.
    (a) Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
    (b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [Hint: Consider the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$]