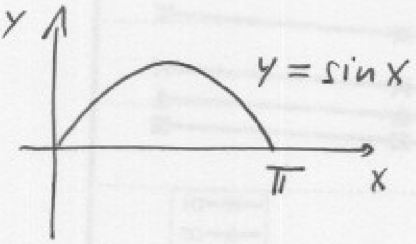
Old Final Exam Questions

>> Solutions <<

1.



<u>mass</u> $$m = \int_0^{\pi}\int_0^{\sin x} k\,dy\,dx = \int_0^{\pi} k \sin x\,dx$$

$$= k(1-\cos\pi) = 2k$$

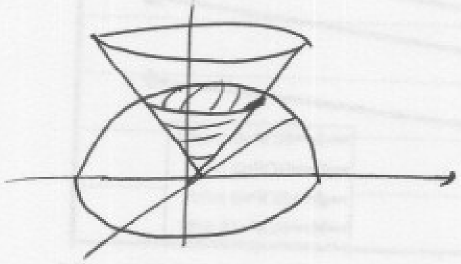
moment of inertia about y-axis

$$I_y = \int_0^{\pi}\int_0^{\sin x} x^2 k\,dy\,dx = \int_0^{\pi} kx^2 \sin x\,dx$$

$$= k\left(-x^2\cos x + 2x\sin x + 2\cos x\right)\Big|_0^{\pi}$$

$$= k(+\pi^2 - 2 + 2) = k(\pi^2 - 4)$$

2.



cone: $\sqrt{3(x^2+y^2)} = z$

sphere: $\sqrt{2-x^2-y^2} = z$

intersection: $\sqrt{3(x^2+y^2)} = \sqrt{2-x^2-y^2}$

$$3x^2+3y^2 = 2-x^2-y^2$$

$$x^2+y^2 = \frac{1}{2} \qquad \text{circle of radius } 2.$$

Use cylindrical coordinates $x = r\cos\theta \quad y = r\sin\theta \quad z = z.$

$$R = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ \sqrt{3}\,r \le z \le \sqrt{2-r^2}\}$$

$$\iiint_R z\,dV = \int_0^{\frac{1}{\sqrt{2}}}\int_0^{2\pi}\int_{\sqrt{3}r}^{\sqrt{2-r^2}} z\, r\,dz\,d\theta\,dr$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} \frac{z^2}{2}\Big|_{\sqrt{3}r}^{\sqrt{2-r^2}} r\,dr$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} r\left(\frac{2-r^2}{2} - \frac{3}{2}r^2\right)dr$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} r + r^3\left(-\frac{1}{2} - \frac{3}{2}\right)dr$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} r - 2r^3\,dr$$

$$= 2\pi\left(\frac{r^2}{2} - \frac{r^4}{2}\Big|_0^{\frac{1}{\sqrt{2}}}\right)$$

$$= \cancel{0} \quad 2\pi\left(\frac{1}{4} - \frac{1}{8}\right) = \frac{\pi}{4}$$

R2

3. $\vec{F} = \langle yz\cos x,\ z\sin x,\ y\sin x + 2z \rangle$

$\vec{r}(t) = \langle \frac{\pi}{2} t^2,\ e^{t^2},\ \cos^4(t\pi) \rangle \quad 0 \le t \le 1$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz\cos x & z\sin x & y\sin x + 2z \end{vmatrix}$$

$$= \vec{i}\,(\sin x - \sin x) + \vec{j}\,(y\cos x - y\cos x)$$

$$+ \vec{k}\,(+z\cos x - z\cos x) = 0$$

$\Rightarrow$ $\vec{F}$ is conservative.

$\Rightarrow$ $\int_C \vec{F}\,d\vec{r}$ is path independent.

$\vec{r}(t)$ connects $t=0: (0, 1, 1)$ with

$t=1: (\frac{\pi}{2}, e, 1)$

Choose another path $\vec{y}(t) = (1-t)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} \frac{\pi}{2} \\ e \\ 1 \end{pmatrix}$

$$\vec{y}(t) = \begin{pmatrix} \frac{\pi t}{2} \\ 1 + (e-1)t \\ 1 \end{pmatrix} \qquad \vec{y}\,'(t) = \begin{pmatrix} \frac{\pi}{2} \\ e-1 \\ 0 \end{pmatrix}$$

$$\vec{F}(\vec{y}(t)) = \left\langle (1+(e-1)t)\cos\frac{\pi t}{2},\ \sin\frac{\pi t}{2},\ (1+(e-1)t)\sin\frac{\pi t}{2} + 2 \right\rangle$$

$$\int_C \vec{F}\, d\vec{r} = \int_0^1 \vec{F}(\vec{y}(t))\, \vec{y}\,'(t)\, dt$$

$$= \int_0^1 \frac{\pi}{2}(1+(e-1)t)\cos\frac{\pi t}{2} + (e-1)\sin\frac{\pi t}{2} \quad dt$$

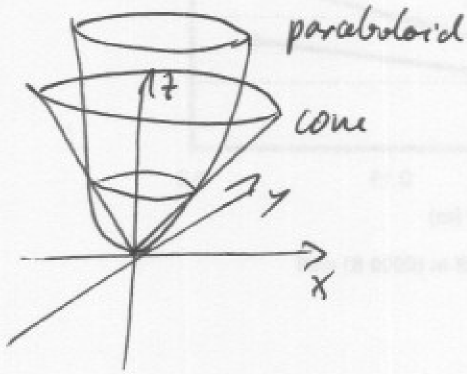
$$= \frac{\pi}{2}\left(1+(e-1)t\right)\frac{2}{\pi}\sin\frac{\pi t}{2}\Big|_0^1 - \frac{\pi}{2}\int_0^1 (e-1)\frac{2}{\pi}\sin\frac{\pi t}{2}\, dt$$

$$+ \int_0^1 (e-1)\sin\frac{\pi t}{2}\, dt$$

$$= 1 + e - 1$$

$$= e$$

4.



Intersection

$$3(x^2+y^2) = (x^2+y^2)^2$$

Use cylindrical coordinates

$$3r^2 = r^4$$

$$3 = r^2 \implies r = \sqrt{3}.$$

cone $\quad z^2 = 3(x^2+y^2) = 3r^2$

$$\vec{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ \sqrt{3}r \rangle$$

$$\vec{r}_r = \langle \cos\theta,\ \sin\theta,\ \sqrt{3} \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \sqrt{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \vec{i}\left(-\sqrt{3}\, r\cos\theta\right) + \vec{j}\left(\sqrt{3}\, r\sin\theta\right) + \vec{k}\left(r\cos^2\theta + r\sin^2\theta\right)$$

$$= r\left(-\sqrt{3}\cos\theta\ \vec{i} + \sqrt{3}\sin\theta\ \vec{j} + \vec{k}\right)$$

$$\|\vec{r}_r \times \vec{r}_\theta\| = r\sqrt{3\cos^2\theta + 3\sin^2\theta + 1}$$

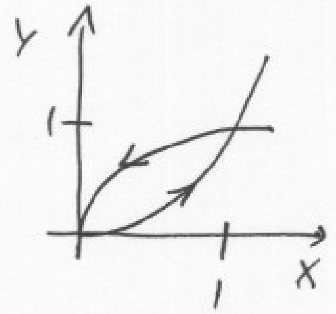
$$= r\sqrt{4} = 2r$$

$$\iint_S x^2+y^2\, dS = \int_0^{\sqrt{3}}\int_0^{2\pi} r^2\, 2r \cdot dr\, d\theta = 2\pi \int_0^{\sqrt{3}} 2r^3\, dr$$

$$= 2\pi \left.\frac{r^4}{2}\right|_0^{\sqrt{3}} = 9\pi$$

5. a) $\int_C (y+e^{\sqrt{x}})\,dx + (2x+\cos(y^2))\,dy$

C bounds the region enclosed by $y=x^2$, $x=y^2$:



Use Greens theorem

$$\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

$Q_x = 2$, $P_y = 1$

$D = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$

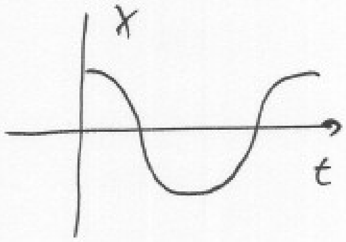
$$\int_C (y+e^{\sqrt{x}})\,dx + (2x+\cos(y^2))\,dy$$

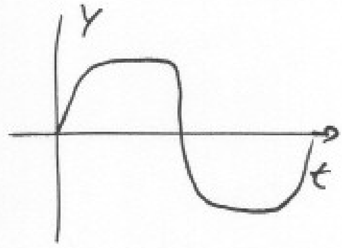
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy\,dx$$

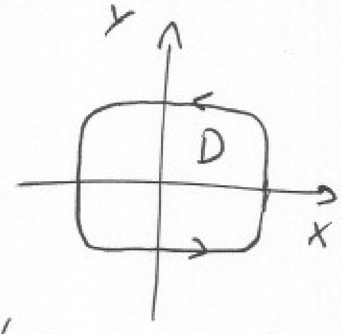
$$= \int_0^1 \sqrt{x} - x^2 \, dx$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

5b) $x = \cos t \quad y = \sin^3 t \quad 0 \le t \le 2\pi$







Use Greens theorem $\iint_D dA = \int_C x\,dy \quad (P=0,\ Q=x)$

$$= \int_0^{2\pi} \cos t \underbrace{3\sin^2 t \cos t}_{y'(t)}\, dt$$

$$= 3\int_0^{2\pi} \cos^2 t \sin^2 t \quad dt$$

$$= 3\int_0^{2\pi} \cos^2 t\,(1-\cos^2 t)\,dt$$

$$= 3\int_0^{2\pi} \cos^2 t - \cos^4 t \,dt$$

$$= 3\int_0^{2\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos 2t - \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right)^2 dt$$

$$= 3\int_0^{2\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos 2t - \tfrac{1}{4} - \tfrac{1}{2}\cos 2t - \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4t\right) dt$$

$$= 3\int_0^{2\pi} \tfrac{1}{8} - \tfrac{1}{8}\cos 4t \,dt$$

$$= \tfrac{3}{4}\pi - \tfrac{3}{8}\,\frac{\sin 4t}{4}\Big|_0^{2\pi} = \underline{\underline{\tfrac{3}{4}\pi}}$$

7. a) $u = \sin(x - at)$

$u_t = -a\cos(x-at)$ $\qquad$ $u_{tt} = -a^2 \sin(x-at)$

$u_x = \cos(x-at)$ $\qquad$ $u_{xx} = -\sin(x-at)$

$u_{tt} = a^2 u_{xx}$ ✓

b) $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$

Tangent plane for an implicitely defined surface

$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$

$F_x = \frac{1}{2}x$ $\qquad$ $F_y = 2y$ $\qquad$ $F_z = \frac{2}{9}z$

$F_x(-2, 2, -3) = -1$

$F_y(-2, 2, -3) = 4$

$F_z(-2, 2, -3) = -\frac{2}{3}$

$-1(x+2) + 4(y-2) - \frac{2}{3}(z+3) = 0$

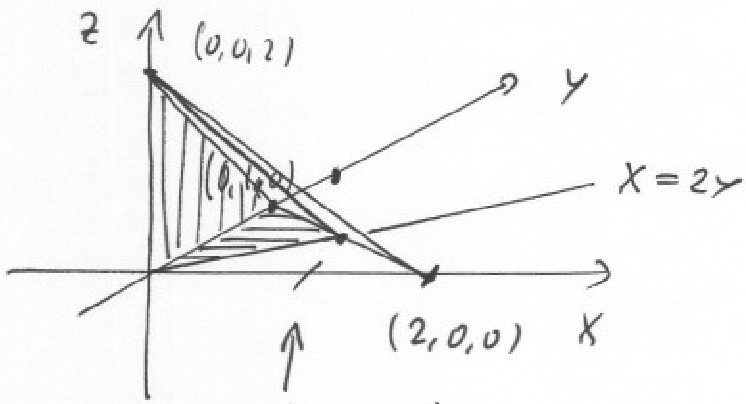
$-x - 2 + 4y - 8 - \frac{2}{3}z + 2 = 0$

$-x + 4y - \frac{2}{3}z = 8$

R8

8. a) tetrahedron: $x + 2y + z = 2$

$x = 2y, \quad z = 0$



need intersection point: $x = 2y$ & $x + 2y = 2$

$y = \frac{1}{2}(2-x)$

$\Rightarrow \quad 2x = 2 \qquad x = 1.$

$E = \{ (x,y,z) \mid 0 \le x \le 1, \ \frac{1}{2}x \le y \le \frac{1}{2}(2-x),$

$0 \le z \le 2 - x - 2y \}$

$$\text{Vol } E = \int_0^1 \int_{\frac{x}{2}}^{\frac{1}{2}(2-x)} \int_0^{2-x-y} dz\,dy\,dx$$

$$= \int_0^1 \int_{\frac{x}{2}}^{\frac{1}{2}(2-x)} (2-x-y)\,dy\,dx$$

$$= \int_0^1 2 - x\left(\frac{1}{2}(2-x) - \frac{x}{2} \right) - \frac{1}{2}\left(\frac{1}{4}(2-x)^2 - \frac{x^2}{4} \right) dx$$

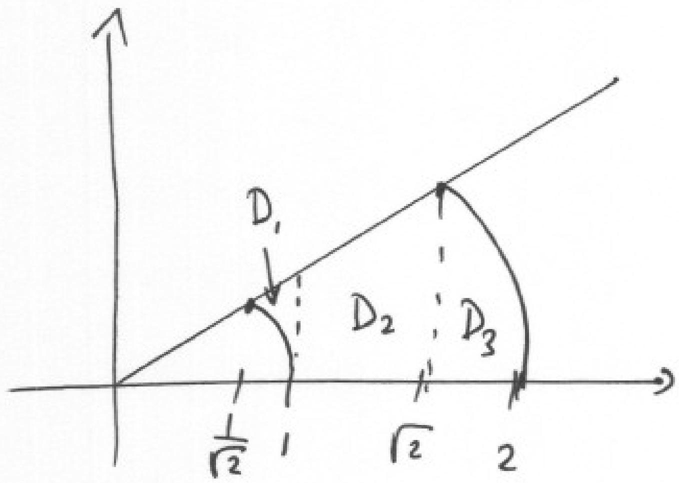
$= \ldots .$

R9

8b) $D_1 = \{ (x,y) : \frac{1}{\sqrt{2}} \le x \le 1 ,\ \sqrt{1-x^2} \le y \le x \}$

$D_2 = \{ (x,y) : 1 \le x \le \sqrt{2} ,\ 0 \le y \le x \}$

$D_3 = \{ (x,y) : \sqrt{2} \le x \le 2 ,\ 0 \le y \le \sqrt{4-x^2} \}$



$D = \{ (r,\theta) : 1 \le r \le 2 ,\ 0 \le \theta \le \frac{\pi}{4} \}$

$$\iint_D xy\, dy\, dx = \int_1^2 \int_0^{\frac{\pi}{4}} r^2 \cos\theta \sin\theta \, r\, dr\, d\theta$$

$$= \frac{r^4}{4} \Big|_1^2 \cdot \frac{1}{2} \sin^2\theta \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{15}{8} \sin^2 \frac{\pi}{4}$$