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Mathematical and Statistical Sciences
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Projects

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Contents

1 Salmon

Let x_n be the number of hundreds of millions of Pacific salmon at the beginning of the n^{th} cycle. They produce a larval population y at a time t_0, which is proportional to the number of adult salmon x_n, with proportionality constant β.

What happens to the larvae? The adults cannabalize them. Then the larval population decays at a rate which is proportional to the number of interactions with the adult population, with proportionality constant α. The larvae do not remain larvae forever, and so this decay occurs only during an interval of time $t_0 < t < t_e$, which is a portion of the cycle. After this, the young adults (the larvae that survived) go out to sea, and a fraction γ of them will survive. The other adults will die after breeding. So the number of salmon at the beginning of the next cycle will be the number of young adults that survive at sea.

Set up the equations which govern the salmon population, and determine if there is an equilibrium solution or cyclical behaviour. Some typical parameter values are $1 < \alpha(t_e - t_0) < 10$ and $3 < \beta\gamma < 20$. How could factors such as fishing, pollution, infections with parasites from fish farms, etc., affect the behaviour of the population?

2 Laser

Consider a particular type of laser known as a solid-state laser, which consists of a collection of special "laser-active" atoms embedded in a solid-state matrix, bounded by partially reflecting mirrors at either end. An external energy source is used to excite or "pump" the atoms out of their ground state (see Figure 1).

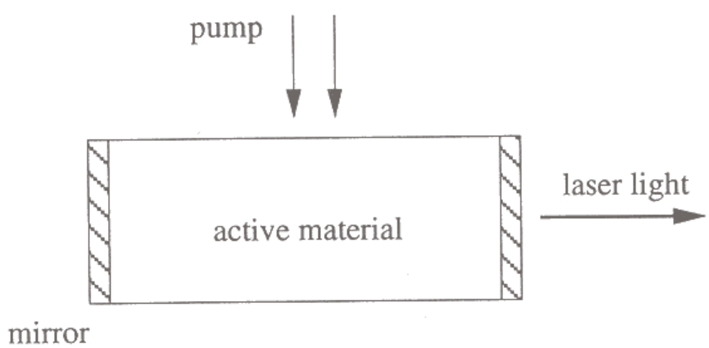


Figure 1: Cartoon representation of a laser.

Each atom can be thought of as a little antenna radiating energy. When the pumping is relatively weak, the laser acts just like an ordinary lamp: the excited atoms oscillate independently of one another and emit randomly phased light waves.

Now suppose that we increase the strength of the pumping. At first, nothing different happens, but then suddenly, when the pump strength exceeds a certain threshold, the atoms begin to oscillate in phase – the lamp has turned into a laser. Now the trillions of little antennas act like one giant antenna and produce a beam of radiation that is much more coherent and intense than that produced below the laser threshold.

This sudden onset of coherence is amazing, considering that the atoms are being excited completely at random by the pump! Hence the process is *self-organizing*: the coherence develops because of a cooperative interaction among the atoms themselves.

A proper explanation of the laser phenomenon would require us to delve into quantum mechanics. Instead, consider a simplified model of the essential physics. The dynamical variable $n(t)$ represents the number of photons in the field. Its rate of change is given by

$$\begin{aligned}\frac{dn}{dt} &= \text{gain} - \text{loss} \\ &= GnN - kn.\end{aligned}$$

The gain term comes from the process of stimulated emission, in which photons stimulate excited atoms to emit additional photons. Because this process occurs via random encounters between photons and excited atoms, it occurs at a rate proportional to n and to the number of excited atoms, denoted by $N(t)$. The parameter $G > 0$ is known as the gain coefficient. The loss term models the escape of photons through the endfaces of the laser. The parameter $k > 0$ is a rate constant; its reciprocal $\tau = 1/k$ represents the typical lifetime of a photon in the laser.
Now comes the key physical idea: after an excited atom emits a photon, it drops down to a lower energy level and is no longer excited. Thus, N decreases by the emission of photons. To capture this effect in the model, we need to write down an equation relating N to n. Suppose that in the absence of laser action, the pump keeps the number of excited atoms fixed at N_0. Then the *actual* number of excited atoms will be reduced by the laser process. Specifically, assume

$$N(t) = N_0 - \alpha n,$$

where $\alpha > 0$ is the rate at which atoms drop back to their ground states. Then the model for n becomes

$$\begin{aligned}\frac{dn}{dt} &= Gn(N_0 - \alpha n) - kn \\ &= (GN_0 - k)n - (\alpha G)n^2.\end{aligned}$$

(a) Analyze the last equation for various values of N_0. How do the number of steady states depend on the value of N_0? Sketch a bifurcation diagram for this model, that is, graph the value of the steady states as a function of N_0. Represent stable steady states by a solid curve, and unstable steady states by a dashed line. How does the laser threshold show up?

Although this model correctly predicts the existence of a threshold, it ignores the dynamics of excited atoms, the existence of spontaneous emission, and several other complications. Two improved models are accessible, and worthy of further investigation.

An improved model of a laser. In the simple model considered above, we wrote an *algebraic* equation relating N to n. In more realistic models, this relation would be replaced by a *differential* equation. For example, it can be shown that after reasonable approximations (if you are interested in the details, references can be provided), quantum mechanics leads to the following system of equations:

$$\begin{aligned}\frac{dn}{dt} &= GnN - kn, \\ \frac{dN}{dt} &= -GnN - fN + p.\end{aligned}$$

(b) Study the quasi-static approximation of this system, that is, suppose that N relaxes much more rapidly than n. Then we can make the assumption that $dN/dt \approx 0$. With this approximation, N can be expressed in terms of n. Eliminating N from the first equation gives a first-order equation for n. This procedure is often called adiabetic elimination, and one says that the evolution of $N(t)$ is slaved to that of $n(t)$. Analyze the resulting equation for n as you analyzed the previous model. How do the results differ?

(c) Study the full system. That is, explore the behaviour of the model with a numerical simulation, plus determine steady states and their stability. Sketch all the qualitatively different phase portraits that occur as the model parameters are varied. Summarize your results in bifurcation diagrams.

The Maxwell-Bloch Equations. The Maxwell-Bloch equations provide an even more sophisticated model for a laser. These equations describe the dynamics of the electric field E, the mean polarization P of the atoms, and the population inversion D:

$$\begin{aligned}\dot{E} &= \kappa(P - E),\\ \dot{P} &= \gamma_1(ED - P),\\ \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP),\end{aligned}$$

where κ is the decay rate in the laser cavity due to beam transitions, γ_1 and γ_2 are decay rates of the atomic polarization and population inversion, respectively, and λ is the pumping energy parameter. The parameter λ may be positive, negative, or zero; all the other parameters are positive.

(d) Create a simulation for the model, and explore the behaviour of the model numerically, using a variety of different parameter sets. What do you observe?

These equations are similar to the famous Lorenz equations, and can exhibit chaotic behaviour for certain parameter sets. However, many practical lasers do not operate in the chaotic regime. In the very simplest case, $\gamma_1, \gamma_2 >> \kappa$, and then P and D relax rapidly to steady values, and hence may be diabetically eliminated.

(e) Assume that $\dot{P} \approx 0$ and $\dot{D} \approx 0$, and express P and D in terms of E, and thereby derive a first-order equation for E. Determine all steady states of the equation for E and determine their stability. Summarize the results in a bifurcation diagram.

3 Tree Harvest

A forester has to decide when to cut trees which have been planted. At a time t, the forester cuts N trees. The selling price is \$5 per cubic foot of wood. The combined planting and harvesting costs are \$50,000 plus \$20 per tree. The volume v of wood (in hundreds of cubic feet) of the tree depends on time, and satisfies the following model for tree growth:

$$\begin{aligned}\frac{dv}{dt} &= k_1 v - k_2 v^2\\ v(0) &= v_0\end{aligned}$$

Some possible values for the model parameters are $v_0 = 0 : 0001, k1 = 1$, and $k2 = 0.1$. Time t is in years.

(a) Understand the model and explain the meaning of the model parameters.

(b) Assuming that the forester would like to plant more trees after the harvest, and harvest them at a time 2t, etc., what would you recommend for the value of t? That is, consider several consecutive harvests.

(c) Suppose you include inflation in your model, so that the costs and prices increase with a factor e^{rt}, with $r > 0$. Also, suppose you discount the final value of the trees (profits - costs) with a factor $e^{-\nu t}$, where $\nu > 0$ is the nominal annual interest rate. This discount factor models the fact that the trees are an investment, so their 'uncut' value is discounted by the amount of interest that could have been earned if they had been harvested and the money invested. Choose reasonable values of r and μ, and show how it affects the optimal time of harvest.

(d) Include the actual political situation and analyze your prediction in the case that our neighbor (USA) restricts lumber imports. How would your strategy change under export constraints?

(e) Consider global warming over time. Global warming might lead to a faster growth rate of the trees, a larger carrying capacity of the ecosystem. Faster growth might change the quality of the wood, which will change the prize. Try to include these effects. Under what conditions is global warming beneficial?

4 Arms Race

In this project, we consider several models for an arms race. Consider two economically competing nations which we call Purple and Green. Both nations desire peace and hope to avoid war, but they are not pacifistic. They will not go out of their way to launch agression, but they will not sit idly by if their country is attacked. They believe in self-defense and will fight to protect their nation and their way of life. Both nations feel that the maintenance of a large army and the stockpiling of weapons are purely "defensive" gestures when they do it, but at least somewhat "offensive" when the other side does it.
Since the two nations are in competition, there is an underlying sense of "mutual fear". The more one nation arms, the more the other nation is spurred to arm.

1. A Mutual Fear Model. Let $x(t)$ and $y(t)$ represent the yearly rates of armament expenditures of the two nations in some standardized monetary unit. To develop a model of mutual fear, we assume that each country adjusts the rate of increase or decrease of its armaments in response to the level of the other's. The simplest assumption is that each nation's rate is directly proportional to the expenditure of the other nation. That is,

$$\begin{aligned} \frac{dx}{dt} &= ay, \\ \frac{dy}{dt} &= bx, \end{aligned}$$

where a and b are positive constants. Suppose the initial (at some arbitrary time $t = 0$) armament expenditures of the two nations are x_0 and y_0, respectively. Find dy/dx in terms of x and y, and solve using the given initial conditions. Show that the solutions lie on a hyperbola, and interpret the results. Verify your solution numerically.

2. The Richardson Model. We now present a refinement of the mutual fear model. The mutual fear model produced a "runaway" arms race with unlimited expenditures. To prevent unlimited expenditures, we assume that excessive armament expenditures present a drag on the nation's economy so that the actual level of expenditure reduces the rate of change on the expenditure. The simplest way to model this is the assume that the rate of change for a nation is directly and negatively proportional to its own expenditure. This refines the mutual fear model to give

$$\begin{aligned} \frac{dx}{dt} &= ay - mx, \\ \frac{dy}{dt} &= bx - ny, \end{aligned}$$

where a, b, m, and n are positive constants.
Before proceeding with a mathematical analysis of this model, we introduce a further refinement. This refinement models any underlying grievances of each country toward the other. To model this, we introduce two additional constant terms, r and s, to the equations to obtain

$$\begin{aligned} \frac{dx}{dt} &= ay - mx + r, \\ \frac{dy}{dt} &= bx - ny + s. \end{aligned}$$

A positive value of r and s indicates that there is a grievance of one country toward the other which causes an increase in the rate of arms expenditures. If r or s is negative, there there is an underlying feeling of good will, so there is a decrease in the rate of arms expenditures. This model is called Richardson's Arms Race Model in honour of Lewis F. Richardson, who considered this model in 1939 for the combatants of World War I.
Use phase plane analysis to investigate the behaviour of the Richardson model (look at the nullclines, determine steady states, and their stability). Focus on the following two scenarios:

(a) **Mutual Grievances.** Investigate the case where each side has a permanent underlying grievance toward the other side. In this case the parameters r and s are positive. Show that the steady state for the model either lies in the first or the third quandrant, depending upon the sign of $mn - ab$. Determine their stability. What does the model predict? Verify numerically.

(b) **The Effect of Good Will.** Feelings of good will are represented by one of the grievance terms r or s being negative. If both r and s are negative, show that the steady state is unstable. Further, show that if $mn - ab$ is positive, the arms race results in total disarmament of both sides. What happens if $mn - ab$ is negative? Verify your results numerically.

3. Open-ended Extensions of the Richardson Model. This part offers some suggested further extensions of the Richardson model. Some of these are much more involved than the work in parts 1 and 2 (you should view parts 1 and 2 as warmup problems for the actual project contained in part 3).
There are several directions one could choose to extend the Richardson model. Here are two. The first extension concerns a modification of the mutual fear term discussed in part 1. If we assume that there is an inherent limit to the amount a nation can spend on armaments and let K_p be the maximum expenditure of the Purple nation and K_g be the maximum expenditure of the Green nation, then the proportionality constant can be replaced by an expenditure-dependent rate. A simple way to do this is to replace exponential factors with logistic factors. Thus our arms race model becomes

$$\begin{aligned} \frac{dx}{dt} &= a\left(1 - \frac{x}{K_p}\right) y - mx + r, \\ \frac{dy}{dt} &= b\left(1 - \frac{y}{K_g}\right) x - ny + s. \end{aligned}$$

Examine the stability of this "logistic" version of the Richardson model by performing an analysis similar to part 2. What does this model predict?
A second extension considers three mutually fearful nations. Each nation is spurred to arm by the expenditures of the other two. Build a Richardson model for three nations. Examine the stability of this model by performing an analysis similar to part 2. This three-nation model can be further modified if two of the nations are close allies who are not threatened by the arms buildup of each other but are threatened by the expenditures of the third.

5 Blood Alcohol

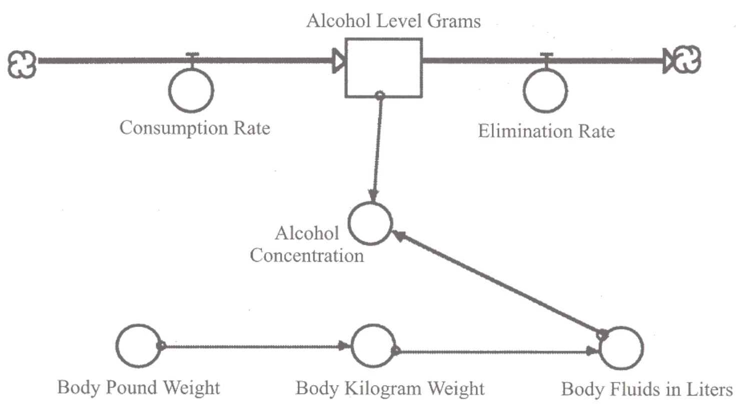
Data:

- The average human body eliminates 12 grams of alcohol per hour.
- An average college-age male in good shape weighing K kilograms has about $0.68K$ liters of fluid in his body. A college-age female in good shape weighing K kilograms has about $0.65K$ liters of fluid in her body. People in poor shape have less.
- One kilogram = 2.2046 pounds.
- Threshold for legal driving: If your body fluids contain more than one gram of alcohol per liter of body fluids (or 0.1 gm/100 mL which is the usual way of reporting it), then you are too drunk to drive legally in most jurisdictions. Find out the level for Alberta, and use it in this project.
- A blood alcohol concentration of 4.0 gm/L is likely to result in coma. A blood alcohol level of 4.5 to 5.0 gm/L is likely to result in death.
- The alcohol content of various beverages is listed Figure 2.

Construct a model for alcohol concentration based on the diagram shown in Figure 2 for a hypothetical person (gender and weight decided by you). You may use a discrete-time model with a short time step (1 minute is suggested), or treat time continuously (ideally, you would do both and compare your results).
Assume that Hypothetical arrives at a party and instantaneously downs a six-pack of beer. If Hypothetical doesn't enter a coma or die, how long will it be before he/she can drive home legally?
Construct a more realistic manner of consuming six beers. How does this affect Hypothetical's blood alcohol concentration? You may wish to use a piecewise defined function to model periods of drinking and non-drinking.
Construct an information sheet for distribution to your peers showing the effects of drinking style on blood alcohol concentration.

6 Run-Bike-Fun

The following Run-Bike-Fun sports event takes place every year in a small university town in Germany. Each participating team consists of two people. Both people have to complete a 15 km course through a combination of running and cycling. Each team has one bicycle. Only one person is allowed to ride the bicycle at any one time, but team members can switch between running and cycling as often as they wish. The first team with both partners at the finish line wins. At the beginning of the race, one person starts riding the bicycle, and the other starts running. After some time, the cyclist gets off the bicycle, puts it down, and starts running.



Type of Drink	Grams of Alcohol
12 ounce regular beer	13.6
12 ounce light beer	11.3
4 ounce port wine	16.4
4 ounce burgundy wine	10.9
4 ounce rose wine	10.0
1.5 ounce 100-proof vodka	16.7
1.5 ounce 100-proof bourbon	16.7
1.5 ounce 80-proof vodka	13.4
1.5 ounce 80-proof bourbon	13.4

Figure 2: Compartmental diagram for the blood alcohol model and grams of alcohol for different types of drink.

When the other runner reaches the bicycle, he/she picks it up, and starts cycling. What is the optimal switching strategy? At which locations along the course should the switch(es) occur? You may wish to begin by assuming that it takes no time to get on/off the bicycle, that both team members are x times faster at cycling compared to running, and that people run/cycle with constant velocity. Based on your own experience, estimate the value of x. When/where should you switch?
In reality, people get tired. How might you describe that? Would you use the same description for running and cycling? How does this affect the optimal strategy? Also, switching between cycling and running takes time. How does this affect the optimal strategy? What if two people with different abilities form a team?

7 Carbon Cycle

One class of environmental models tracks the flow of some element or compound through an ecosystem. This project examines the flow of carbon through several types of forest conditions.

Part 1. A Litter Model. We begin with a very simple model, tracking carbon levels in litter on a forest floor. In this context, litter is naturally-occurring debris such as leaves, branches, and deadfalls, not beer bottles or fast-food wrappers. A boundary for the system is set up, and only litter within this region is considered. Thus, carbon is measured in density units: grams of carbon per square meter (g C/m^2). Here are the modelling assumptions:

1. Carbon continuously enters the system through litterfall at a constant rate z.
2. Carbon continuously leaves the system through two avenues – carbon dioxide produced in respiration and the conversion of litter into humus (called humification). Even though the carbon in humus has not left the physical system boundary, it is no longer in the litter, just as carbon in the trees before the leaves fell was in the physical boundary, but not yet in the litter.
3. The rate of litterfall is constant; the rate of carbon removal from both avenues is proportional to the amount of carbon present.
4. Initially, there is no carbon in the litter. This approximates the situation after a ground fire (a forest fire which burns the underbrush, but does not kill the trees).

Set up a differential equation for the carbon level, and solve it analytically. Verify numerically. For a temperate forest, it is reasonable to assume a rate of litterfall of 240 g C/m^2/yr and a proportionality constant for carbon removal of 0.4/ yr. Using this information, graph a solution curve for 50 years either by numerically solving or by graphing the analytical solution using these parameters.

Part 2. Carbon in the Terrestial Biosphere. This model extends the model from part 1 to model carbon's flowing and being stored in an entire ecosystem. This ecosystem is assumed to have seven components: plants (subdivided into leaves, branches, stems, and roots), litter lying on the ground, humus, and stable humus charcoal. The amount of carbon in each of these components is given by the variables x_1, x_2, …, x_7, respectively. The atmosphere is, of course, another component, but due to its immense size, it is considered to have constant carbon content, unchanged either by giving carbon to plants or by absorbing carbon from the litter, humus, or stable humus charcoal. Technically the atmosphere is outside the system, so carbon to plants is carbon entering the system, and carbon out of the litter/soil components is carbon leaving the system. The parameter z denotes the carbon entering the system, and the partition parameters p_1 through p_4 indicate the percentage of z which goes into the leaves, branches, stems, and roots, respectively. The transfer coefficients k_{ij} give the rate of carbon flow from x_i to x_j (k_{i0} will denote the transfer from x_i to the atmosphere). A compartmental diagram for this system and model parameters for a variety of ecosystem types are shown in Figure 3. Tropical forest refers to tropical forest, forest plantation, shrub-dominated savannas, and chaparral. Temperate forests comprise temperate forests, boreal forests, and woodlands. Note that k_{56} and k_{k0} are not given, only an outflow of carbon from litter. As in part 1, carbon is lost from the litter state by either humification or respiration. The humification factor h gives the fraction of carbon leaving the litter state that goes into humus; naturally, $1 - h$ indicates the fraction of carbon leaving the litter that does not go to humus, and hence goes to the atmosphere by respiration. Similar comments apply to k_{67} and k_{60} involving the amount of carbon leaving the humus and the coefficient of carbonization c which is the fraction going to form stable humus charcoal. The unit Gt is a Gigatonnes, or a billion tonnes, where 1 tonne (metric ton) $= 1,000$ kg $= 1$ Mg.

(a) Use the compartmental diagram to set up the system of equations for this model.

(b) For this part, use the parameters for the tropical forest ecosystem. First find the fixed points of this system. To do this, first write the answer to part (a) in the matrix form $X' = AX + B$. Fixed points occur when $X' = 0$, so solve the matrix equation $AX + B = 0$ (Maple will come in handy here). Numerically solve the system to get graphs of each variable as a function of time. Run the simulation until all values are within 95% of their stable values. Compare the relative stabilization time for each component (how long it takes to get within 95% of the stable value). Compute the eigenvalues for the fixed point. Can you see a relationship between the size of the eigenvalues and the stabilization times?

(c) Repeat (b) for at least two other ecosystems (your choice) and compare the results.

8 Yellow Fever in Senegal

Yellow fever (YF) is a viral hoemorrhagic fever transmitted by infected mosquitoes. Yellow fever is spread into human populations in three stages:

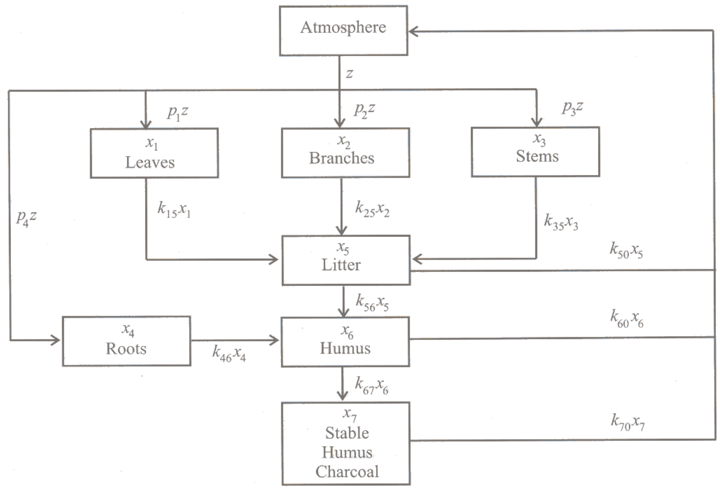
1. **Sylvatic** (or jungle) YF occurs in tropical rain forests where mosquitoes, which feed on infected monkeys, pass the virus to humans who work in the forest.

2. **Intermediate** YF occurs as infected individuals bring the disease into rural villages, where it is spread by mosquitoes amongst humans (and also monkeys).

3. **Urban** YF occurs as soon as an infected individual enters urban areas. This can lead to an explosive epidemic in densely inhabited regions. Domestic mosquitoes carry the virus from person to person.

The epidemic can be controlled by vaccination. YF vaccine is safe and effective and provides immunity within one week in 95 % of those vaccinated.

Table **??** shows a data set of YF cases and YF deaths during an outbreak in Senegal in 2002, collected from the internet archives of the World Health Organization (WHO). As soon as the virus was identified, a vaccination program was started (Oct 1, 2002). On Oct 11, 2002 the disease was reported in Touba, a city of 800,000 residents. More information can be found on the WHO websites.



	Tropical forest	Temperate forest	Grass-land	Agri-cultural area	Human area	Tundra and semi-desert
Carbon entering System z (Gt C/yr)	27.8	8.7	10.7	7.5	0.2	2.1
Partition coefficients						
p_1 (Leaves)	0.3	0.3	0.6	0.8	0.3	0.5
p_2 (Branches)	0.2	0.2	0.0	0.0	0.2	0.1
p_3 (Stems)	0.3	0.3	0.0	0.0	0.3	0.1
p_4 (Roots)	0.2	0.2	0.4	0.2	0.2	0.3
Flows						
Leaves to litter	1.0	0.5	1.0	1.0	1.0	1.0
Branches to litter	0.1	0.1	0.1	0.1	0.1	0.1
Stems to litter	0.033	0.0166	0.02	0.02	0.02	0.02
Roots to humus	0.1	0.1	1.0	1.0	0.1	0.5
Leaving litter	1.0	0.5	0.5	1.0	0.5	0.5
Leaving humus	0.1	0.02	0.025	0.04	0.02	0.02
Charcoal to atmosphere	0.002	0.002	0.002	0.002	0.002	0.002
Humification h	0.4	0.6	0.6	0.2	0.5	0.6
Carbonization c	0.05	0.05	0.05	0.05	0.05	0.05
Areas ($10^{12}m^2$)	36.1	17.0	18.8	17.4	2.0	29.7

Figure 3: The terrestial carbon system and parameters of carbon storage and flow in terrestial ecosystems.

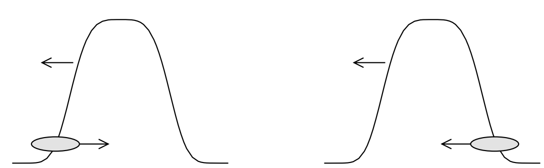
report date	cases (total)	deaths (total)
Jan 18th	18	0
Oct 4th	12	0
Oct 11th	15	2
Oct 17th	18	2
Oct 24th	41	4
Oct 31st	45	4
Nov 20th	57	10
Nov 28th	60	11

Yellow Fever in Senegal, 2002 (data from the disease outbreak news archives of the WHO

1. Develop a model for the three stages of YF as outlined above.
2. Include a fourth stage which describes vaccination in urban areas.
3. Fit your model to the data.
4. What would have happened without vaccination?
5. Would you expect that the disease dies out, or that it becomes persistent?

9 The Chemotactic Paradox

A variety of mechanisms have evolved by which living systems sense the environment in which they reside and respond to signals they detect, often by changing their patterns of movement. The movement response can entail changing the speed of movement and the frequency of turning, which is called *kinesis*, it may involve direct movement, which is called *taxis*, or it may involve a combination of these. Taxis and kinesis may be characterized as positive or negative, depending on whether they lead to accumulation at high or low concentrations of an external stimulus. Typical stimuli for microorganisms are light, gravitation, pressure, or some chemical signal. Tactic and kinetic responses both involve the detection of the external signal and transduction of this signal into an internal signal that triggers the response.
An example of chemotaxis occurs in *Dictyostelium discoideum*, where individuals aggregate in response to a signal from "organizers" cells. Individual cells relay the signal to their neighbours, thereby causing an outward moving travelling wave of the chemical through a Dictyostelium population. We are interested in the movement of a single cell as it responds to the moving signal pulse.



Sketch of the chemotactic paradox.

As a model case, we assume that the wave is moving in one spatial dimension from right to left, as shown in the above Figure. A cell at the wave front senses an increasing signal concentration and moves forward (to the right), opposite to the direction of the wave. As soon as the wave back passes the cell, the cell senses a negative gradient of the signal concentration. Hence it should move backwards (to the left), which is in the direction of the wave. Overall, the cell would spend more time in the wave back than in the wave front, which should give a net displacement to the left. In experiments, however, cells move to the right only. This is called the *chemotactic paradox*. Formulate a hypothesis to resolve the paradox and investigate your hypothesis with the use of a mathematical model. Can you resolve the paradox?

10 Don't Lek the Frogs

(Thanks to Barry Bohnet)
A "lek" is a gathering of males during mating season where the males display certain traits that the females of the species are attracted to. This occurs in many species including some deer, grouse, peacocks, fish, and many others. The attractive trait could be as simple as bright colours or as elegant as a song. According to Fishers sexy-son hypothesis, this results in most of the females mating with a select group of males. The effects of this selection result in a population bottleneck which is a reduction of gene flow.
DNA is the simplest unit of heredity. Most animals are diploid organisms, meaning that each individual contains two copies of chromosome. These chromosomes are long strands of DNA made up of genes. Each gene can come in a variety of forms, where each of these forms is called an allele. In the sex cells that each parent passes on to its child, there is one of the two chromosome copies. In other words each parent passes on 50% of its genes. A genotype refers to the pair of alleles that an individual has. G. H. Hardy and Wilhelm Weinberg came up with a formula known as the Hardy-Weinberg Principle:

$$p^2 + 2pq + q^2 = 1$$

Where p and q are the frequencies of two alleles, of one gene, present in the current generation. The frequencies p and q must add up to 1, since they are the only alleles of a certain gene. This formula gives us genotypic frequencies of the following generation.
Using the Hardy-Weinberg Principle we shall model the diversity of a bird species that uses leks in its mating patterns, and determine what type of effect this has on genotypic frequencies, what causes stable fixed points, and where they occur.

11 Marriage

At what age are your friends going to marry? at 18? 20? 25? 30? What factors cause people to marry? Sociaologists and psychologists generally believe that peer group behavior plays a role. Can you model this? The following attempt is adapted from G. Hernes (1972):

1. It is assumed that a person chances of marrying in some small time interval Δt are proportional to Δt and to the fraction of people in the person's age group which are already married $m(t)$. This is based on the idea that there is overt and covert peer group pressure to marry. Show that this leads to the differential equation

   $$m' = cm(1 - m)$$

   Solve this equation.

2. Try to find data from Statistics Canada to parametrize your model.

3. The model can be criticized for various reasons. For example, it assumes that all people feel the same pressure to marry, regardless of individual and age as long as the fraction of the peer group that is married is the same. Discuss the model critically.

4. Suppose $c = c(t)$. How can this make the model more realistic? What is the solution to the differential equation in this case? In terms of $c(t)$ determine what fraction of people in your age class will eventually marry.

5. Hernes finds that
$$\log(c(t)) = ab^t \log k, \quad b < 1$$
gives a rather good fit for $c(t)$. Sketch the above function and also $c(t)$ and explain what properites a realistic function $c(t)$ should have.

6. We forgot to discuss the problem, that initially zero people in your peer group will be married. Hence, by the above equation, $m(t)$ would stay zero at all times. How can we get around this?

7. Discuss how to include the effect that people are not identical. Can this be incorporated into the model somehow?

8. Hillen: In my hometown in Germany there is a custom that a 30 year old unmarried male has to sweep in front of the city hall, as a sign to all young ladies that here is a 30 year old fellow who is not yet married. He has to sweep until an unmarried women comes along and relieves him with a kiss. Can you include this in your model?

12 Gonorrhea

In this problem we study models for gonorrhea epidemics. Gonorrhea is spread by sexual intercourse, takes 3 to 7 days to incubate, and can be cured by the use of antibiotics. There is no evidence that a person ever develops immunity.

1. Let x be the fraction of men who are infected and let f be the fraction of men who are promiscuous. Let y and g be the corresponding quantities for women. Discuss this model:
$$\begin{aligned} x' &= -ax + b(f - x)y \\ y' &= -cy + d(g - y)x, \end{aligned}$$
where a, b, c, d are constants. Interpret the model and the constants.

2. What are the equilibrium points of this model? Which ones are stable? Provide phase plane sketches. You should find that the number $(a/bf)(c/dg)$ is critical. Under what conditions on the parameters will there be an epidemic?

3. Interpret and discuss strategies to change the frequency of sexual intercourse, the fraction of the population that is promiscuous (of either sex), and the speed of cure.

4. What advice would you give health officials to manage an gonorrhea outbreak in Canada, or in Hong Kong?

5. Develop a model like the above for homosexual males. Analyse this and give advice to the health officials related to the homosexual core group.

6. How would the model for homosexuals change if you want to model HIV transmission. This is an open ended question and feel free to speculate a bit.

13 Hockey Coach

Assume you are a primer league hockey coach. Of course your goal is to assemble the best team possible. However, the owner can provide only a finite amount of funds to purchase players. You are faced with the challenge to hire high-level players who are all reasonably good, or to hire some few super stars but also some mediocre players.

1. The hockey league keeps statistics about each relevant or irrelevant information on players. Based on your experience and a web-search, determine relative quality measures for your players. First use very few quality measures, later use more.
2. Design a mathematical model which translates these individual measures into a quality measure for the team. Compare your quality measure with the performance history of a few teams.
3. With this model come back to the original question. Is it better to spend lots of money on super-stars, or is it better to hire a good homogeneous team?
4. Include the effect of your strategy on your fans. A super star will draw more people into the arena and your revenue from tickets, merchandise and advertisement goes up.
5. Consider extra motivation of the players for bonuses for goals or other achievements.
6. Let me know when you can start working for the Oilers.

14 Whooping Crane

The whooping crane population which is breading in the Wood Buffalo National Park has been counted for the past several years. The numbers for the years 1938 until 2003 are

$$18, 22, 26, 16, 19, 21, 18, 22, 25, 31, 30, 34, 31, 25, 21, 24, 21, 28, 24, 26, 32, 33, 36, 39, 32, 33,$$
$$42, 44, 43, 48, 50, 56, 57, 59, 51, 49, 49, 57, 69, 72, 75, 76, 78, 73, 73, 75, 86, 97, 110, 134, 138,$$
$$146, 146, 132, 136, 143, 133, 158, 160, 180, 183, 188, 180, 176, 185, 194$$

1. Use the discrete Ricker model to fit the data:

   $$x_{n+1} = a x_n e^{-r x_n}$$

   Which time unit do you choose? You might consider a transformation of $\ln(x_{n+1}/x_n)$ and use a least squares linear fit. What are the best fit values for a and r?
2. For general parameters a and r fully analyse the discrete Ricker model: Find equilibrium points and their stability.
3. Can you find periodic orbits, period doubling, and chaos ? Use cobwebbing to illustrate your findings.
4. Plot a Feigenbaum diagram.
5. Compare the general analysis to the fitted model from above. Under what conditions would you expect chaotic behavior in the whooping crane population? Is this an indication of a healthy population or not?
6. Try to include declining resources for the crane over time, due to human interference. Would we loose the cranes at some point?

15 Modelling with Graphs

Graph theory can be a useful tool for modelling as well. I will not be able to cover graph theory in class, but I would like to give one of you the opportunity to present it to the rest of us.

Please review pages 289-296 from our textbook and explain the graph theoretical approach. Also solve the coloring problems for Australia (problems 3 and 5) and the final exam scheduling problem (problem 7).

16 Queuing Problems

Queuing problems are another class of problems which are not covered in class. Please work your way through the Harbour queuing problem and do problems 1,2,3,4 on pages 204-212.