MATH 300 Fall 2007
Advanced Boundary Value Problems I

Practice Problems for Final Examination

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Question 1. Given the function

$$f(x) = x, \quad -\pi < x < \pi$$

find the Fourier series for f and use Dirichlet's convergence theorem to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin na}{n} = \frac{a}{2}$$

for $0 < a < \pi$.

Question 2. Let $0 < a < \pi$, given the function

$$f(x) = \begin{cases} \dfrac{1}{2a} & \text{if} \quad |x| < a \\ 0 & \text{if} \quad x \in (-\pi, \pi], \quad \text{and} \quad |x| > a \end{cases}$$

find the Fourier series for f and use Dirichlet's convergence theorem to show that

$$\sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{1}{2}(\pi - a)$$

for $0 < a < \pi$.

Question 3. Consider the regular Sturm-Liouville problem

$$(x\phi')' + \lambda^2 \frac{1}{x}\phi = 0 \qquad 1 \le x \le 2$$

$$\phi(1) = 0$$

$$\phi(2) = 0$$

(a) The general solution to the differential equation is

$$\phi(x) = A\cos(\lambda \ln x) + B\sin(\lambda \ln x).$$

Find the eigenvalues λ_n^2 and the corresponding eigenfunctions ϕ_n for this problem.

(b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(c) Use the Rayleigh quotient to estimate the smallest eigenvalue of this regular Sturm-Liouville problem. **Note:** From part (a), the first eigenvalue and eigenfunction are

$$\lambda_1^2 = \left(\frac{\pi}{\ln 2}\right) \approx 20.5423 \qquad \text{and} \qquad \phi_1(x) = \sin\left(\frac{\pi \ln x}{\ln 2}\right).$$

Try to find a reasonable estimate.

Question 4. Find the solution of the **exterior Dirichlet problem for a disk**, that is find a bounded solution to the problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \qquad a<r<\infty, \quad -\pi<\theta<\pi$$

$$u(r,\pi)=u(r,-\pi) \qquad a<r<\infty$$

$$\frac{\partial u}{\partial \theta}(r,\pi)=\frac{\partial u}{\partial \theta}(r,-\pi) \quad a<r<\infty$$

$$u(a,\theta)=f(\theta) \qquad -\pi<\theta<\pi.$$

Question 5. Find all functions ϕ for which $u(x,t)=\phi(x+ct)$ is a solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{k}\frac{\partial u}{\partial t}$$

where k and c are constants.

Question 6. Consider torsional oscillations of a homogeneous cylindrical shaft. If $\omega(x,t)$ is the angular displacement at time t of the cross section at x, then

$$\frac{\partial^2 \omega}{\partial t^2}=a^2\frac{\partial^2 \omega}{\partial x^2} \qquad 0<x<L, \qquad t>0.$$

Solve this problem if

$$\omega(x,0)=f(x) \qquad 0<x<L$$

$$\frac{\partial \omega}{\partial t}(x,0)=0 \qquad 0<x<L,$$

and the ends of the shaft are fixed elastically:

$$\frac{\partial \omega}{\partial x}(0,t)-\alpha\,\omega(0,t)=0 \quad t>0$$

$$\frac{\partial \omega}{\partial x}(L,t)+\alpha\,\omega(L,t)=0 \quad t>0$$

with α a positive constant.

Question 7. Use D'Alembert's solution of the wave equation to solve the initial value - boundary value problem:

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad -\infty<x<\infty, \qquad t>0$$

$$u(x,0)=f(x) \qquad -\infty<x<\infty$$

$$\frac{\partial u}{\partial t}(x,0)=g(x) \qquad -\infty<x<\infty$$

with $f(x)=0$ and $g(x)=\dfrac{x}{1+x^2}$.

Question 8. Obtain the expansion

$$e^{ax}=\frac{\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx-n\sin nx)$$

valid for all real numbers $a\neq 0$, and all $-\pi<x<\pi$.

Question 9. Consider the regular Sturm-Liouville problem

$$\left(x^2 X'\right)' + \lambda X = 0 \quad 1 < x < e$$
$$X(1) = 0$$
$$X(e) = 0$$

(a) Show that the substitution

$$X = \frac{Y}{\sqrt{x}}$$

transforms this problem into the following problem

$$x\left(xY'\right)' + \mu Y = 0 \quad 1 < x < e$$
$$Y(1) = 0$$
$$Y(e) = 0$$

where $\mu = \lambda - \dfrac{1}{4}$.

(b) Let $x = e^t$ and $\widehat{Y}(t) = Y(e^t)$, show that this transforms the problem in part (a) into the problem

$$\frac{d^2\widehat{Y}}{dt^2} + \mu \widehat{Y} = 0 \quad 0 < t < 1$$
$$\widehat{Y}(0) = 0$$
$$\widehat{Y}(1) = 0.$$

(c) Find the eigenvalues and eigenvectors for the problem in part (b), and from these, the eigenvalues and eigenfunctions in part (a), and finally obtain the eigenvalues and eigenfunctions for the original Sturm-Liouville problem.

Question 10. Show that if $|a| < 1$, then

(a) $\displaystyle\sum_{n=1}^{\infty} a^n \cos nx = \frac{a\cos x - a^2}{1 - 2a\cos x + a^2}$ for $-\pi < x < \pi$,

(b) $\displaystyle\sum_{n=1}^{\infty} a^n \sin nx = \frac{a\sin x}{1 - 2a\cos x + a^2}$ for $-\pi < x < \pi$,

Question 11. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 - \cos x & \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Question 12. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} x & \text{if} \quad -1 < x < 1, \\ 2 - x & \text{if} \quad 1 < x < 2, \\ -2 - x & \text{if} \quad -2 < x < -1, \\ 0 & \text{otherwise.} \end{cases}$$

Question 13. Let

$$f(x) = \begin{cases} x & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Plot the function $f(x)$ and find its Fourier transform.

(b) If $\widehat{f}$ is real valued, plot it; otherwise plot $|\widehat{f}|$.

Question 14. Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1 - x & \text{if} \quad 0 < x < 1, \\ 0 & \text{if} \quad x \geq 1. \end{cases}$$

and write $f(x)$ as an inverse cosine transform. Use a known Fourier transform and the fact that if $f(x),\ x \geq 0$, is the restriction of an *even* function f_e, then

$$\mathcal{F}_c(f)(\omega) = 2\mathcal{F}(f_e)(\omega)$$

for all $\omega \geq 0$.

Question 15. Find the Fourier sine transform of

$$f(x) = \frac{x}{1 + x^2}, \qquad x > 0,$$

and write $f(x)$ as an inverse sine transform. Use a known Fourier transform and the fact that if $f(x),\ x \geq 0$, is the restriction of an *odd* function f_o, then

$$\mathcal{F}_s(f)(\omega) = -2i\mathcal{F}(f_o)(\omega)$$

for all $\omega \geq 0$.

Question 16. Use the Fourier transform to solve the heat flow problem in an infinite rod

$$\frac{\partial u}{\partial t} = 10\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = \begin{cases} 2 & \text{for} \quad -\pi \leq x \leq \pi \\ 0 & \text{otherwise,} \end{cases}$$

and express the solution as the difference of two error functions.

Question 17. Find the solution to the initial value problem

$$\frac{\partial u}{\partial t} + 5\frac{\partial u}{\partial x} = e^{3t}, \ -\infty < x < \infty, \ t \geq 0$$

$$u(x, 0) = e^{-x^2}, \qquad -\infty < x < \infty$$

using the method of characteristics.