Solutions to assignment # 5

8.2.1 e)) equilibrium satisfies $\frac{d}{dt}v(x) = 0$, hence

$$\left.\begin{array}{l} k v_{xx} = -k \\ v_x(0) = 0 \\ v_x(L) = 0 \end{array}\right\} \Rightarrow \quad v(x) = -\frac{x^2}{2} + ax + b$$

$$v_x(x) = -x + a$$

$$v_x(0) = 0 = a \Rightarrow a = 0$$

$$v_x(L) = -L \neq 0 \Rightarrow \text{there exists no equilibrium.}$$

To find a problem with homogeneous boundary values, I don't need to do anything, since it has homogeneous b.c. already. But to make the problem homogeneous I can write

$$u(x,t) = w(x,t) + kt \quad , \text{ which gives}$$

$$\left.\begin{array}{l} w_t = k w_{xx} \\ w_x(0) = 0 = w_x(L) \end{array}\right\}.$$

8.2.2.b)) $$\left.\begin{array}{l} u_t = k u_{xx} + Q(x,t) \\ u(x,0) = f(x) \\ u(0,t) = A(t) \\ u_x(L,t) = B(t) \end{array}\right\}$$

try: $u(x,t) = v(x,t) + w(x,t)$.

$$v_t + w_t = k v_{xx} + k w_{xx} + Q \qquad \Bigg| \quad v(0,t) + w(0,t) = A(t)$$

$$v(x,0) + w(x,0) = f(x) \qquad \Bigg| \quad v_x(L,t) + w_x(L,t) = B(t)$$

v-problem

$v(0,t) = A(t)$

$v_x(L,t) = B(t)$

w-problem

$w_t = k w_{xx} + k v_{xx} - v_t + Q$

$w(x,0) = f(x) - v(x,0)$

$w(0,t) = 0$

$w_x(L,t) = 0$

Solve the v-problem: $v(x,t) = A(t) + B(t)x$

Thus the remaining problem with homogeneous boundary conditions is

$$\left.\begin{aligned} w_t &= k w_{xx} - \dot{A}(t) - \dot{B}(t)x + Q(x,t) \\ w(x,0) &= f(x) - A(0) - B(0)x \\ w(0,t) &= 0 \\ w_x(L,t) &= 0 \end{aligned}\right\}$$

8.2.3|

Step 1: Make homogeneous boundary conditions:

$$\left.\begin{aligned} u(r,t) &= v(r) + w(r,t) \\ w_t &= \frac{k}{r}\left(\frac{\partial}{\partial r}\left(r\,\frac{\partial(v+w)}{\partial r}\right)\right) + Q(r) \\ v(r) + w(r,0) &= f(r) \\ v(a) + w(a,t) &= T \end{aligned}\right\}$$

$\underline{v\text{-problem}}$ $\qquad\qquad\qquad\qquad$ $\underline{w\text{-problem}}$

$$\left.\begin{aligned} &\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)=0 \\ &v(a)=T \end{aligned}\right\} \qquad \left.\begin{aligned} &w_t=\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+Q \\ &w(r,0)=f(r)-v(r) \\ &w(a,t)=0 \end{aligned}\right\}$$

Solve v-problem: $v(r)=T$ $\qquad$ (constant).

Then $\quad u(r,t)=w(r,t)+T$ $\quad$ and

$$\left.\begin{aligned} &w_t=\frac{k}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right)+Q(r) \\ &w(r,0)=f(r)-T \\ &w(a,t)=0 \end{aligned}\right\} \quad (*)$$

Step 2

look at homogeneous problem:

$$\left.\begin{aligned} &w_t=\frac{k}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right) \\ &w(a,t)=0, \\ &|w(0,t)|<\infty \end{aligned}\right\}$$

Separation: $w(r,t)=R(r)\,H(t)$

$$RH'=\frac{k}{r}\left((rR')'\right)H$$

$$\frac{H'}{kH}=\frac{1}{rR}(rR')'=-\lambda$$

$$H'=-\lambda k H,$$

$$r(rR')'+\lambda r^2R=0, \quad R(a)=0, \ |R(0)|<\infty$$

The R-problem is a Bessel equation of order $m=0$. The solution is $J_0(\sqrt{\lambda_{0n}}\, r)$, where λ_{0n} are given as $\sqrt{\lambda_{0n}}\, a = z_{0n}$, which are the zeros of $J_0(z)$.

<u>Step 3</u> To solve (*) we assume

$$w(r,t) = \sum_{n=1}^{\infty} a_n(t)\, J_0(\sqrt{\lambda_{0n}}\, r).$$

We need the Bessel series of $Q(r)$ and of $f(r)-T$:

$$Q(r) = \sum_{n=1}^{\infty} q_n J_0(\sqrt{\lambda_{0n}}\, r) \qquad q_n = \frac{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r)\, Q(r)\, r\,dr}{\int_0^a J_0^2(\sqrt{\lambda_{0n}}\, r)\, r\,dr}$$

$$f(r)-T = \sum_{n=1}^{\infty} f_n J_0(\sqrt{\lambda_{0n}}\, r) \qquad f_n = \frac{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r)\,(f(r)-T)\, r\,dr}{\int_0^a J_0^2(\sqrt{\lambda_{0n}}\, r)\, r\,dr}$$

Substitute into (*):

$$\sum_{n=1}^{\infty} \dot{a}_n(t) J_0(\sqrt{\lambda_{0n}}\, r) = \sum_{n=1}^{\infty} a_n(t)(-\lambda_{0n})\, J_0(\sqrt{\lambda_{0n}}\, r)$$
$$+ \sum_{n=1}^{\infty} q_n J_0(\sqrt{\lambda_{0n}}\, r)$$

Compare coefficients

$$\dot{a}_n(t) = -\lambda_{0n}\, a_n(t) + q_n$$

$$a_n(t) = a_n(0)\, e^{-\lambda_{0n} t} + \int_0^t e^{-\lambda_{0n}(t-s)}\, q_n \, ds$$

$$= a_n(0)\, e^{-\lambda_{0n} t} + \frac{q_n}{\lambda_{0n}}\left(1 - e^{-\lambda_{0n} t}\right)$$

At $t=0$: $$w(r,0) = \sum_{n=1}^{\infty} a_n(0)\, J_0\left(\sqrt{\lambda_{0n}}\, r\right)$$

$$= f(r) - T$$

$$= \sum_{n=1}^{\infty} f_n\, J_0\left(\sqrt{\lambda_{0n}}\, r\right)$$

$$\Longrightarrow \quad a_n(0) = f_n .$$

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Step 4 Put everything together:

$$u(r,t) = T + \sum_{n=1}^{\infty}\left[f_n e^{-\lambda_{0n} t} - \frac{q_n}{\lambda_{0n}} e^{-\lambda_{0n} t} + \frac{q_n}{\lambda_{0n}} \right] J_0\left(\sqrt{\lambda_{0n}}\, r\right)$$

$$= T + \sum_{n=1}^{\infty}\left(\frac{q_n}{\lambda_{0n}} + \left(f_n - \frac{q_n}{\lambda_{0n}}\right) e^{-\lambda_{0n} t} \right) J_0\left(\sqrt{\lambda_{0n}}\, r\right)$$

Sol 5.5

8.3.2 Homogeneous problem

$$\left.\begin{aligned} u_t &= k u_{xx} \\ u(0,t) &= 0 = u(L,t) \end{aligned}\right\} \begin{aligned} &\text{has eigenvalues } \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ &\text{and eigenfunctions } \phi_n = \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

Hence we try: $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$.

We will need the same series expansions for Q and f:

$$Q(x) = \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi x}{L}\right)$$

$$q_n = \frac{2}{\pi} \int_0^L Q(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f_n = \frac{2}{\pi} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

From the non-homogeneous problem we obtain

$$\sum_{n=1}^{\infty} a_n'(t) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} a_n(t)\, k \left(-\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi x}{L}\right)$$

Compare coefficients

$$a_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 a_n + q_n$$

with initial condition $a_n(0) = f_n$.

The solution is

$$a_n(t) = f_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} + q_n \int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)} ds$$

$$a_n(t) = f_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \frac{g_n}{k\left(\frac{n\pi}{L}\right)^2}\left(1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right)$$

$$= \frac{g_n}{k\left(\frac{n\pi}{L}\right)^2} + e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(f_n - \frac{g_n}{k\left(\frac{n\pi}{L}\right)^2}\right)$$

$$\longrightarrow \frac{g_n}{k\left(\frac{n\pi}{L}\right)^2} \text{ for } t \to \infty.$$

Solution

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{g_n}{k\left(\frac{n\pi}{L}\right)^2} + e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(f_n - \frac{g_n}{k\left(\frac{n\pi}{L}\right)^2}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$\xrightarrow[t\to\infty]{} \underbrace{\sum_{n=1}^{\infty}\frac{g_n}{k\left(\frac{n\pi}{L}\right)^2}\sin\left(\frac{n\pi x}{L}\right)}_{=: r(x)} \text{ which is the steady state.}$$

10.3.11 a) $\mathcal{F}(c_1 f + c_2 g) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(c_1 f + c_2 g)e^{i\omega x}\,dx$

$$= \frac{c_1}{2\pi}\int_{-\infty}^{+\infty} f e^{i\omega x}\,dx + \frac{c_2}{2\pi}\int_{-\infty}^{\infty} g e^{i\omega x}\,dx$$

$$= c_1\mathcal{F}(f) + c_2\mathcal{F}(g)$$

b) $\mathcal{F}^{-1}(F(\omega)G(\omega)) = \frac{1}{2\pi} f * g \neq f\cdot g.$

$\boxed{10.3.5}$ $\quad f(x), \quad \hat{f}(\omega) = \mathcal{F}(f(x))$

$$\mathcal{F}^{-1}\left(e^{i\omega\beta}\hat{f}(\omega)\right) = \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega\beta} e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{-i\omega(x-\beta)}\, d\omega$$

$$= f(x-\beta)$$

$\boxed{10.4.4}$

$$\left.\begin{aligned} u_t &= k u_{xx} - \gamma u \qquad -\infty < x < \infty \\ u(x,0) &= f(x) \end{aligned}\right\}$$

call $\hat{u}(\omega,t) = \mathcal{F}(u(x,t))$

$$\hat{u}_t = -k\omega^2 \hat{u} - \gamma \hat{u}$$

$$\hat{u}_t = (-k\omega^2 - \gamma)\hat{u} \implies \hat{u}(\omega,t) = \hat{f}(\omega) e^{-(k\omega^2+\gamma)t}$$

$$\hat{u}(\omega,0) = \hat{f}(\omega)$$

$$u(x,t) = \mathcal{F}^{-1}\left(\hat{u}(\omega,t)\right) = \mathcal{F}^{-1}\left(\hat{f}(\omega) e^{-(k\omega^2+\gamma)t}\right)$$

$$= \mathcal{F}^{-1}\left(\hat{f}(\omega) e^{-k\omega^2 t} e^{-\gamma t}\right) \qquad \mathcal{F}^{-1} \text{ is linear:}$$

$$= e^{-\gamma t}\, \mathcal{F}^{-1}\left(\hat{f}(\omega) e^{-k\omega^2 t}\right)$$

$$= e^{-\gamma t} \frac{1}{2\pi} f * g$$

5\.
Sol 6.7

$$\Longrightarrow \quad u(x,t) = e^{-\gamma t} \int_{-\infty}^{+\infty} f(y) \, \frac{e^{-\frac{(x-y)^2}{4kt}}}{\sqrt{4\pi k t}} \, dy$$

We find the transformation if we multiply by $e^{\gamma t}$.

$e^{\gamma t} u(x,t) = f * G$ looks like a solution of a homogeneous heat equation. Indeed, if we define $w(x,t) := e^{\gamma t} u(x,t)$ we get

$$
\begin{aligned}
w_t &= \gamma e^{\gamma t} u + e^{\gamma t} u_t \\
&= \gamma w + e^{\gamma t} (k u_{xx} \mp \gamma u) \\
&= \gamma w + k w_{xx} - \gamma w
\end{aligned}
$$

hence

$$
\left.
\begin{aligned}
& w_t = k w_{xx} \qquad -\infty < x < \infty \\
& w(x,0) = f(x)
\end{aligned}
\right\}
$$

Sol 6.8

10.3.5 $\hat f(\omega) = \mathcal{F}(f(x))$

$$\mathcal{F}^{-1}\left(e^{i\omega\beta}\hat f(\omega)\right) = \int_{-\infty}^{+\infty} \hat f(\omega)\, e^{i\omega\beta} e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \hat f(\omega)\, e^{-i\omega(x-\beta)}\, d\omega$$

$$= f(x-\beta)$$

10.4.4. $u_t = k u_{xx} - \gamma u \qquad -\infty < x < \infty$

$u(x,0) = f(x)$

a) Use F-transform: $\hat u(\omega,t) = \mathcal{F}(u(x,t))$

$$\mathcal{F}(u_t) = \hat u_t \qquad \mathcal{F}(u_{xx}) = -\omega^2 \hat u$$

$$\hat u_t = -k\omega^2 \hat u - \gamma \hat u$$

$$= -(k\omega^2 + \gamma)\hat u$$

which is solved by

$$\hat u(\omega,t) = \hat u(\omega,0)\, e^{-(k\omega^2+\gamma)t}$$

$$\hat u(\omega,0) = \hat f(\omega)$$

hence

$$\hat u(\omega,t) = \hat f(\omega)\, e^{-(k\omega^2+\gamma)t}$$

$$u(x,t) = \mathcal{F}^{-1}\left(\hat{f}(\omega) e^{-(k\omega^2+\gamma)t} \right)$$

$$= \mathcal{F}^{-1}\left(\hat{f}(\omega) e^{-k\omega^2 t} e^{-\gamma t} \right)$$

$$= e^{-\gamma t} \underbrace{\mathcal{F}^{-1}\left(\hat{f}(\omega) e^{-k\omega^2 t} \right)}_{\text{solution of heat equation without } \gamma.}$$

$$= e^{-\gamma t} \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} f(y) e^{-\frac{(x-y)^2}{4kt}} \, dy$$

b) We multiply by $e^{\gamma t}$:

$$e^{\gamma t} u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} f(y) e^{-\frac{(x-y)^2}{4kt}} \, dy \,,$$

which looks like a solution of the heat eq.

Indeed if we transform $w(x,t) = e^{\gamma t} u(x,t)$ then we get

$$\frac{\partial}{\partial t} w(x,t) = \gamma e^{\gamma t} u + e^{\gamma t} u_t$$

$$= \gamma e^{\gamma t} u + e^{\gamma t} (k u_{xx} - \gamma u)$$

$$= k e^{\gamma t} u_{xx}$$

$$= k w_{xx}$$

$\underline{10.5.12}$ Solve $\quad u_t = k u_{xx} \qquad x > 0$

$$\left.\begin{aligned} u_t &= k u_{xx} \quad x>0 \\ u_x(0,t) &= 0 \\ u(x,0) &= f(x) \end{aligned}\right\}$$

Use cosine-transform: $\tilde{u}(\omega, t) = C(u(x,t))$

$$C(u_t) = \tilde{u}_t$$

$$C(u_{xx}) = -\frac{2}{\pi} \underbrace{u_x(0,t)}_{=0} - \omega^2 \tilde{u}$$

from boundary condition.

$$\Rightarrow \quad \tilde{u}_t = -k\omega^2 \tilde{u}$$

$$\tilde{u}(\omega, t) = \tilde{u}(\omega, 0)\, e^{-k\omega^2 t}$$

$$= \tilde{f}(\omega) e^{-k\omega^2 t} \qquad \tilde{f}(\omega) = C(f(x)).$$

$$= \frac{2}{\pi} \int_0^\infty f(x) \cos \omega x \, dx$$

$$\Rightarrow \quad u(x,t) = \int_0^\infty \tilde{f}(\omega) e^{-k\omega^2 t} \cos \omega x \, d\omega$$

even even even even in ω.

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{-k\omega^2 t} (\cos \omega x - i \sin \omega x)\, d\omega$$

$$= \int_{-\infty}^{+\infty} \frac{\tilde{f}(\omega)}{2} e^{-k\omega^2 t} e^{-i\omega x} \, d\omega$$

$$= \mathcal{F}^{-1}\left(\frac{\tilde{f}(\omega)}{2} e^{-k\omega^2 t} \right) \qquad (*)$$

5.
Sol 6.11

$$\frac{\hat{f}(\omega)}{2} = \frac{1}{2\pi} 2\int_0^{\infty} f(x)\cos(\omega x)\,dx \qquad \text{use even extension}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_{even}(x)\left(\cos(\omega x) + i\sin\omega x\right)dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_{even}(x)\, e^{i\omega x}\,dx$$

$$= \mathcal{F}\left(f_{even}(x)\right) \qquad (**)$$

From $(*)$ and $(**)$ it follows that $u(x,t)$ is a solution of

$$\left.\begin{aligned} &u_t = k u_{xx} \quad -\infty < x < \infty \\ &u(x,0) = f_{even}(x) \end{aligned}\right\}$$

Hence $u(x,t) = f_{even} * G$

$$= \frac{1}{\sqrt{4\pi k t}}\int_{-\infty}^{+\infty} f_{even}(y)\, e^{-\frac{(x-y)^2}{4kt}}\,dy$$

$$= \frac{1}{\sqrt{4\pi k t}}\left\{\int_0^{\infty} f(y)\left(e^{-\frac{(x+y)^2}{4kt}} + e^{-\frac{(x-y)^2}{4kt}}\right)dy\right.$$

Sol 6.12