Solutions to Assignment #4

5.6.1 a) $p=1,\ q=-x^2,\ \sigma=1$

Testfunction: $\phi(x)=ax^2+bx+c$ needs to satisfy the boundary conditions $\phi'(x)=2ax+b \quad \phi'(0)=0=b$

$\phi(1)=0=a+c \Rightarrow \quad c=-a \quad \Rightarrow \phi(x)=a(x^2-1).$

Just choose $a=1$ and compute the R-quotient.

$$-p\phi\phi'\Big|_0^1 = -\phi(1)\phi'(1)+\phi(0)\phi'(0)=0\,, \quad \phi(x)=x^2-1$$

$$R(\phi)=\frac{\int_0^1 \phi'^2+x^2\phi^2\,dx}{\int_0^1 \phi^2\,dx} \cancel{= \frac{\int_0^1 4x^2+x^4-x^2\,dx}{\int_0^1 x}}$$

$$=\frac{\int_0^1 4x^2+x^2(x^4-2x^2+1)\,dx}{\int_0^1 x^4-2x^2+1\,dx}$$

$$=\frac{\int_0^1 4x^2+x^6-2x^4+x^2\,dx}{\int_0^1 x^4-2x^2+1\,dx}$$

$$=\frac{\frac{4}{3}+\frac{1}{7}-\frac{2}{5}+\frac{1}{3}}{\frac{1}{5}-\frac{2}{3}+\frac{1}{2}} = \frac{\frac{140+15-42+35}{105}}{\frac{6-20+15}{30}}$$

$$=\frac{148/105}{1/30}=\frac{148/7}{1/2}=\frac{296}{2}$$

5.6.1 b) $p=1,\ q=-x,\ \sigma=1$

Testfunction $\phi(x)=ax^2+bx+c$, $\phi'(x)=2ax+b$

$\phi'(0)=0=b$ $\qquad \phi'(1)+2\phi(1)=0=2a+2(a+c)$

$\Rightarrow c=-2a$. Choose $\phi(x)=x^2-2$., $\phi'(x)=2x$

$$-p\phi\phi'\Big|_0^1=-\phi(1)\phi'(1)+\phi(0)\phi'(0)=+2\phi^2(1)$$

$$=2$$

$$R(\phi)=\frac{2+\int_0^1 \phi'^2+x\phi^2dx}{\int_0^1\phi^2dx}=\frac{2+\int_0^1 4x^2+x(x^4-4x^2+4)dx}{\int_0^1 x^4-4x^2+4\,dx}$$

$$=\frac{2+\int_0^1 4x^2+x^5-4x^3+4x\,dx}{\int_0^1 x^4-4x^2+4\,dx}$$

$$=\frac{2+\frac{4}{3}+\frac{1}{6}-1+2}{\frac{1}{5}-\frac{4}{3}+4}=\frac{\frac{18+8+1}{6}}{\frac{60+3-20}{15}}=\frac{27/6}{43/15}$$

$$=\frac{27/2}{43/5}=\frac{135}{86}$$

5.8.6| Separation $u(x,t) = \phi(x) g(t)$ leads to

$$\left.\begin{array}{l} \phi'' = -\lambda \phi \\ \phi'(0) - h\phi(0) = 0 \\ \phi'(L) = 0 \end{array}\right\} \text{ and } \qquad g'' = -c^2 \lambda g$$

(a) Use Rayleigh quotient to show $\lambda > 0$:

$$-p\phi\phi'\big|_0^L = -\phi(L)\phi'(L) + \phi(0)\phi'(0) = h\phi^2(0) \geq 0$$

$$q = 0 \geq 0 \implies \lambda \geq 0.$$

If $\lambda = 0$ then $\phi'(x) = 0 \implies \phi(x) = \text{const} = c_0$.

From the boundary condition $\phi'(0) - h\phi(0)$ we find $c_0 = 0$. Hence $\lambda = 0$ is <u>not</u> possible. $\lambda > 0$.

For $\lambda > 0$ we solve $\phi(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$

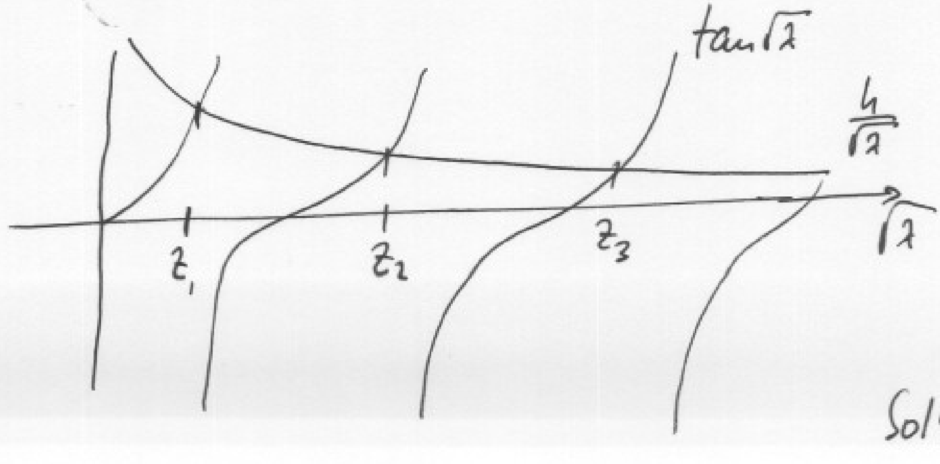
$$\phi'(x) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}x) + b\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\phi'(0) - h\phi(0) = b\sqrt{\lambda} - ha = 0 \implies a = \frac{b}{h}\sqrt{\lambda}$$

$$\phi'(L) = 0 = -\frac{b}{h}\sqrt{\lambda}^2 \sin(\sqrt{\lambda}L) + b\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

divide by $b\sqrt{\lambda}$: $$\frac{\sin(\sqrt{\lambda}L)}{\cos(\sqrt{\lambda}L)} = \frac{h}{\sqrt{\lambda}}$$

$$\tan\sqrt{\lambda} = \frac{h}{\sqrt{\lambda}}$$



(b) for n large : $z_n \sim (2n-1)\frac{\pi}{2}$

$$\Rightarrow \quad \lambda_n \sim \left(\frac{2n-1}{2}\right)^2 \pi^2$$ large frequencies of oscillation.

Note: Most people mean the eigenvalues λ_n as frequencies of oscillations, but not that the frequency of oscillation comes from the solutions of the time problem $h_n(t) = a_n \cos(c\sqrt{\lambda_n}\, t) + b_n \sin(c\sqrt{\lambda_n}\, t)$

One period T corresponds to $c\sqrt{\lambda_n}\, T = 2\pi$

hence $T = \frac{2\pi}{c\sqrt{\lambda_n}}$ and the frequency of oscillation

is $\nu = \frac{1}{T} = \frac{c\sqrt{\lambda_n}}{2\pi} \sim \frac{c}{2\pi}(2n-1)\frac{\pi}{2}$ for large n.

(c) Solution of the ϕ-problem is

$$\phi_n(x) = \frac{\sqrt{\lambda_n}}{h}\cos(\sqrt{\lambda_n}\, x) + \sin(\sqrt{\lambda_n}\, x)$$

which are orthogonal, since the ϕ-problem is a regular SL-problem.

Superposition: $u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(c\sqrt{\lambda_n}\, t) + b_n \sin(\sqrt{\lambda_n}\, t)\right]\phi_n(x)$

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

$$g(x) = \sum_{n=1}^{\infty} b_n \sqrt{\lambda_n}\,\phi_n(x) \qquad b_n = \frac{\int_0^L g(x)\phi_n(x)\,dx}{\sqrt{\lambda_n}\int_0^L \phi_n^2(x)\,dx}$$

5.8.8 (a) $-p\phi\phi'\Big|_0^1 = -\phi(1)\phi'(1) + \phi(0)\phi'(0)$

$$= \phi^2(1) + \phi^2(0) \geq 0$$

$q = 0 \geq 0$. Hence $\lambda \geq 0$. Assume $\lambda = 0$, in that case $\phi^2(1) + \phi^2(0) + \int_0^1 \phi'^2 \, dx = 0$.

Hence $\phi' = 0$ which implies $\phi = \text{constant}$. We also see that $\phi^2(1) = \phi^2(0) = 0$ hence $\phi(x) = 0$, which cannot be an eigenfunction. Hence $\lambda = 0$ is not possible and we conclude $\lambda > 0$.

(b) For a proof you would use the identity $\int_0^1 uLv - vLu\,dx = 0$, see page 179. We can as well use the Mega theorem and say: "The problem is a regular SL-problem, hence it has orthogonal eigenvalues"

(c) Solve $\phi'' = -\lambda\phi$. I know already $\lambda > 0$, hence only one case needed:

$$\phi(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x)$$

$$\phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

$$\phi(0) - \phi'(0) = 0 = c_1 - c_2\sqrt{\lambda} \implies c_1 = c_2\sqrt{\lambda}$$

$$\phi(1) + \phi'(1) = 0 = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}) + c_2\sin(\sqrt{\lambda}) - c_2\sqrt{\lambda}^2\sin(\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda})$$

divide by $c_2 \neq 0$.

$$\frac{\sin(\sqrt{\lambda})}{\cos(\sqrt{\lambda})} = \frac{-2\sqrt{\lambda}}{1-\lambda} \quad \Rightarrow \quad \tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}$$

Discuss the function on the right hand side:

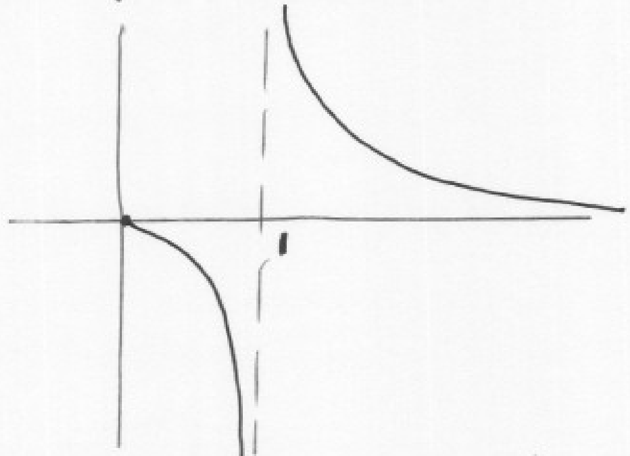
$$f(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{\lambda-1}$$

$f(0) = 0, \quad f(+\infty) = 0$

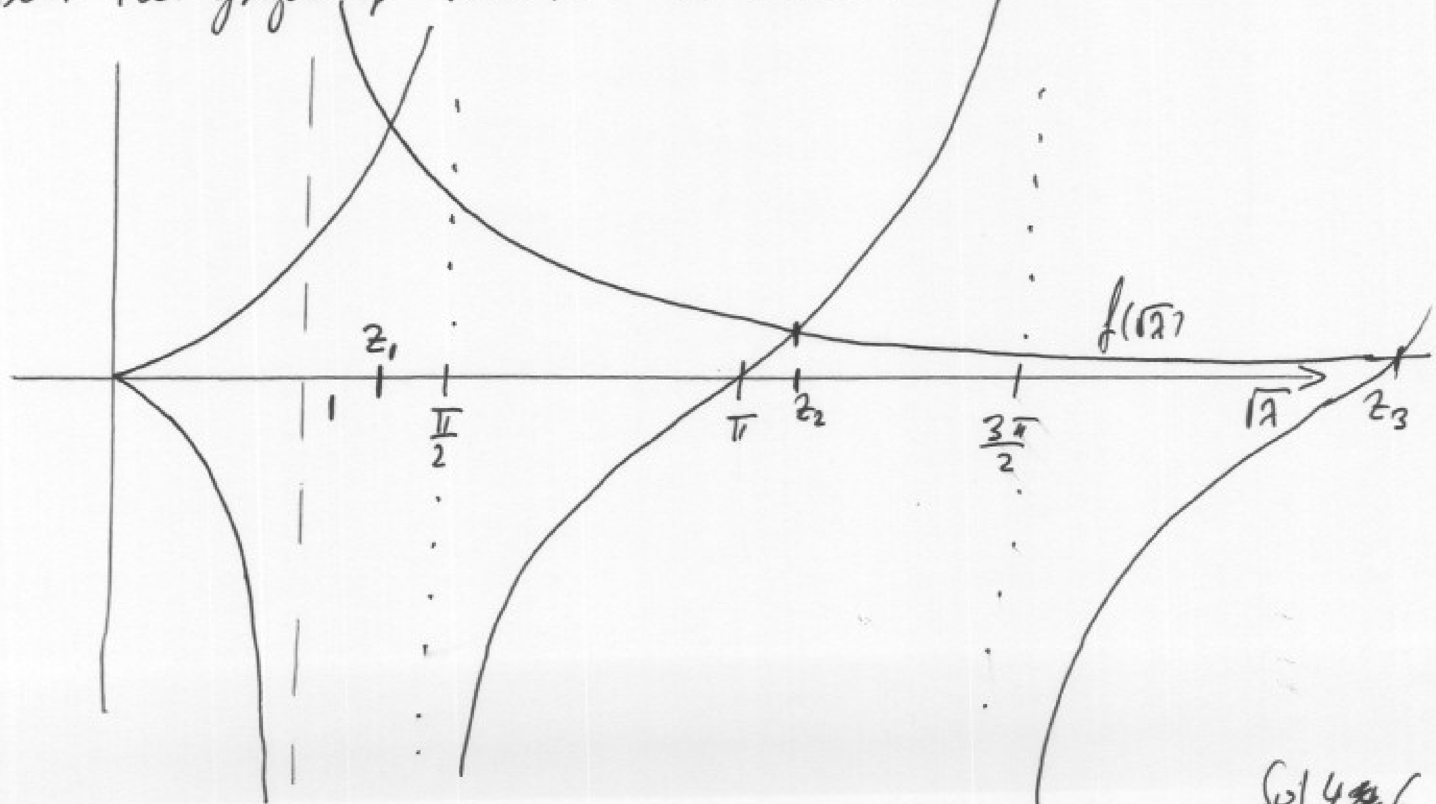
Vertical asymptote at $\lambda = 1$

If $\lambda = 1$ then $f(\sqrt{\lambda}) < 0$

Hence $f(\sqrt{\lambda})$ looks like



Plot the graph of $\tan\sqrt{\lambda}$ as well:



There is an infinite ~~series~~ sequence of intersections

$$\sqrt{\lambda} = z_n \qquad n = 1, 2, 3, \dots$$

which asymptotically behave like $z_n = \text{odd}\ \frac{\pi}{2}$ for large n: $\quad z_n \sim (2n-1)\frac{\pi}{2}$

$$\lambda_n \sim \left(\frac{2n-1}{2}\right)^2 \pi^2 \quad \text{for large } n.$$

(d) Better find the eigenfunction from (c) first: Since $c_1 = c_2\sqrt{\lambda}$ we can write

$$\phi_n(x) = \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\,x) + \sin(\sqrt{\lambda_n}\,x).$$

These are orthogonal from part (b)!

Now use separation: $u(x,t) = h(t)\,\phi(x)$. The $\phi(x)$ problem is now solved (see above). The time problem is solved by $h(t) = c e^{-k\lambda_n t}$.

Superposition gives

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t}\,\phi_n(x)$$

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

$$a_n = \frac{\int_0^1 f(x)\,\phi_n(x)\,dx}{\int_0^1 \phi_n^2(x)\,dx}$$

7.3.1 (b) Separation $u(x,y,t) = X(x)Y(y)T(t)$

$$T'XY = k(X''YT + Y''XT)$$

$$\frac{T'}{kT} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda \quad \text{a first constant}$$

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\tau \quad \text{another constant}$$

$$\left.\begin{array}{l} X'' = -\tau X \\ X'(0) = 0, \; X'(L) = 0 \end{array}\right\} \begin{array}{l} \text{has eigenvalues } \tau_n = \left(\frac{n\pi}{L}\right)^2 \\ n = 0, 1, 2, 3 \ldots \text{ and} \\ \text{eigenfunctions } X_n(x) = \cos\left(\frac{n\pi}{L}x\right) \end{array}$$

$$\left.\begin{array}{l} Y'' = -\alpha Y, \quad \alpha = (\tau - \lambda) \\ Y'(0) = 0, \; Y'(H) = 0 \end{array}\right\} \begin{array}{l} \text{has eigenvalues } \alpha_m = \left(\frac{m\pi}{H}\right)^2 \\ m = 0, 1, 2, \ldots \text{ and} \\ \text{eigenfunctions } Y_m(y) = \cos\left(\frac{m\pi}{H}y\right) \end{array}$$

The combined eigenvalues are $\lambda = \tau + \alpha$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \qquad \begin{array}{l} n = 0, 1, 2, \ldots \\ m = 0, 1, 2, \ldots \end{array}$$

and the combined eigenfunctions are

$$X_n(x)Y_m(y) = \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{H}y\right)$$

The time problem is solved by $T_{nm}(t) = C_{nm}e^{-k\lambda_{nm}t}$

Superposition:

$$u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} e^{-k\lambda_{nm} t} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\sum_{m=0}^{\infty} C_{nm} \cos\left(\frac{m\pi y}{H}\right)\right)}_{B_n(y)} \cos\left(\frac{n\pi x}{L}\right)$$

$$B_0(y) = \frac{1}{L}\int_0^L f(x,y)\,dx$$

$$B_n(y) = \frac{2}{L}\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n \geq 1$$

Then $$B_0(y) = \sum_{m=0}^{\infty} C_{0m} \cos\left(\frac{m\pi y}{H}\right)$$

$$\Rightarrow \quad C_{00} = \frac{1}{H}\int_0^H B_0(y)\,dy = \frac{1}{LH}\int_0^H\int_0^L f(x,y)\,dx\,dy$$

$m \geq 1$ $$C_{0m} = \frac{2}{H}\int_0^H B_0(y)\cos\left(\frac{m\pi y}{H}\right)dy = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{m\pi y}{H}\right)dx\,dy$$

and $$B_n(y) = \sum_{m=0}^{\infty} C_{nm}\cos\left(\frac{m\pi y}{H}\right)$$

$$C_{n0} = \frac{1}{H}\int_0^H B_n(y)\,dy = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)dx\,dy \qquad n \geq 1$$

$$C_{nm} = \frac{2}{H}\int_0^H B_n(y)\cos\left(\frac{m\pi y}{H}\right)dy = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy$$

Sol 4.9

For $n, m \geq 1$ all terms go to zero like $e^{-k\lambda_{mn} t}$.

For $t \to \infty$ only the $n=0$, $m=0$ term remains:

$$u(r, \theta, t) \xrightarrow[t \to \infty]{} C_{00} = \frac{1}{LH} \int_0^H \int_0^L f(x,y)\,dx\,dy$$

which is the total heat energy in our system.

7.7.2a) $u_{tt} = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u\right)$

Separation: $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

$$\frac{T''}{c^2T} = \frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right) + \frac{1}{r^2\Theta}\frac{\partial^2}{\partial\theta^2}\Theta = -\lambda$$

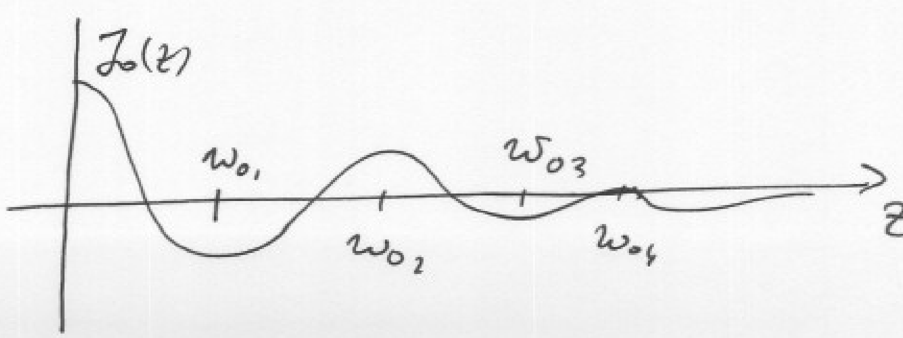
$$\frac{\Theta''}{\Theta} = -\lambda r^2 - \frac{r}{R}(rR')' = -\tau$$

θ-problem: $\left.\begin{array}{l}\Theta'' = -\tau\Theta\\ \Theta(-\pi)=\Theta(\pi)\\ \Theta'(-\pi)=\Theta'(\pi)\end{array}\right\}$ eigenvalues $\tau = m^2$ $m=0,1,2,\dots$ eigenfunctions $\Theta_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta)$

r-problem: $\left.\begin{array}{l} r(rR')' + (\lambda r^2 - m^2)R = 0\\ R'(a)=0, \quad R(0)<\infty\end{array}\right\}$ is a Bessel equation of order m with $z=\sqrt{\lambda}\,a$.

The condition $|R(0)|<\infty$ excludes the Bessel functions of second kind $Y_m(z)$. The condition $R'(a)=0$ leads to

$J_m'(\sqrt{\lambda}\,a) = 0$. Let w_{mn} denote the minima and maxima of $J_m(z)$ in increasing order



Then $\sqrt{\lambda_{mn}}\, a = w_{mn}$ or $\lambda_{mn} = \left(\frac{w_{mn}}{a}\right)^2$.

In particular $\lambda_{mn} > 0$. and $R_{mn} = J_m(\sqrt{\lambda_{mn}}\, r)$

The time problem is solved by

$$T_{mn}(t) = a_{mn} \cos(c\sqrt{\lambda_{mn}}\, t) + b_{mn} \sin(c\sqrt{\lambda_{mn}}\, t)$$

Summarize:

θ	R	T
$\cos(m\theta)$	$J_m(\sqrt{\lambda_{mn}}\, r)$	$\cos(c\sqrt{\lambda_{mn}}\, t)$
$\sin(m\theta)$		$\sin(c\sqrt{\lambda_{mn}}\, t)$
	$m = 0, 1, 2, \ldots$	
	$n = 1, 2, 3, \ldots$	

Superposition

$$u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \cos(m\theta) \cos(c\sqrt{\lambda_{mn}}\, t)$$

$$+ b_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \cos(m\theta) \sin(c\sqrt{\lambda_{mn}}\, t)$$

$$+ C_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin(m\theta) \sin(c\sqrt{\lambda_{mn}}\, t)$$

$$+ d_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin(m\theta) \cos(c\sqrt{\lambda_{mn}}\, t)$$

$u(r,\theta,0) = 0 \Rightarrow a_{mn} = 0, \quad d_{mn} = 0$

$$u_t(r,\theta,0) = \beta(r)\cos(5\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{mn} c\sqrt{\lambda_{mn}}\, J_m(\sqrt{\lambda_{mn}}\, r) \cos(m\theta)$$

$$+ C_{mn} c\sqrt{\lambda_{mn}}\, J_m(\sqrt{\lambda_{mn}}\, r) \sin(m\theta)$$

$\Rightarrow \quad c_{mn} = 0 \quad , \quad b_{mn} = 0 \quad \text{for} \quad m \neq 5$

and $$\sum_{n=1}^{\infty} b_{5n}\, c\sqrt{\lambda_{5n}}\, J_5(\sqrt{\lambda_{5n}}\, r) = \beta(r)$$

$$\Rightarrow \quad b_{5n} = \frac{1}{c\sqrt{\lambda_{5n}}} \; \frac{\int_0^a J_5(\sqrt{\lambda_{5n}}\, r)\, \beta(r)\, r\, dr}{\int_0^a J_5^2(\sqrt{\lambda_{5n}}\, r)\, r\, dr}$$

and
$$u(r,\theta,t) = \sum_{n=1}^{\infty} b_{5n}\, J_5(\sqrt{\lambda_{5n}}\, r)\, \sin(c\sqrt{\lambda_{5n}}\, t)\, \cos(5\theta)$$

<u>7.7.4 b)</u> Separation leads to: $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

Θ-problem:
$$\left.\begin{array}{l} \Theta'' = -\tau\,\Theta \\ \Theta(0) = 0, \quad \Theta(\frac{\pi}{3}) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \tau_m = (3m)^2 \quad m = 1,2,\dots \\ \Theta_m = \sin(3m\theta) \end{array}$$

r-Problem:
$$\left.\begin{array}{l} r(rR')' + (\lambda r^2 - 9m^2)R = 0 \\ R(a) = 0, \quad |R(0)| < \infty \end{array}\right\}$$

which is the Bessel equation of order $3m$, hence solutions are $R_m = J_{3m}(\sqrt{\lambda}\, r)$. $\lambda_{3mn} = \left(\frac{z_{3mn}}{a}\right)^2$

t-Problem: $T(t) = a\cos(c\sqrt{\lambda_{3mn}}\, t) + b\sin(c\sqrt{\lambda_{3mn}}\, t)$.

Frequency ν: $$\frac{c\sqrt{\lambda_{3mn}}}{\nu} = 2\pi \quad \Rightarrow \quad \nu = \frac{c\sqrt{\lambda_{3mn}}}{2\pi} = \frac{c\, z_{3mn}}{2\pi a}$$

Sol 4.12

$$f(r,\theta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin(m\theta)$$

$$\sum_{n=1}^{\infty} a_{mn} J_m(\sqrt{\lambda_{mn}}\, r) = \frac{2}{\pi} \int_0^{\pi} f(r,\theta) \sin(m\theta)\, d\theta$$

$$a_{mn} = \frac{\int_0^a J_m(\sqrt{\lambda_{mn}}\, r)\, \frac{2}{\pi} \int_0^{\pi} f(r,\theta) \sin(m\theta)\, d\theta\, r\, dr}{\int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\, dr}$$

Since $\lambda_{mn} > 0$ we find that $u(r,\theta,t) \longrightarrow 0$ for $t \to \infty$.