Solutions to Assignment #3

4.2.1 Assume $\rho_0(x) = \text{const.}$

a) $$T_0 u_{xx} - g\rho_0 = 0$$

$$u_{xx} = \frac{g\rho_0}{T_0}$$

$$\Rightarrow \quad u_E(x) = \frac{g\rho_0}{T_0} x^2 + C_1 x + C_2$$

$u_E(0) = 0 \Rightarrow C_2 = 0$, $\quad u_E(L) = 0 \Rightarrow \frac{g\rho_0}{T_0} L^2 + C_1 L = 0$

$$\Rightarrow \quad C_1 = -\frac{g\rho_0 L}{T_0}$$

Hence $$u_E(x) = \frac{g\rho_0}{T_0}\left(x^2 - Lx\right)$$

b) $v(x,t) = u(x,t) - u_E(x)$

$v_{tt} = u_{tt}$

$$v_{xx} = u_{xx} - u_{E\,xx} = u_{xx} - \frac{g\rho_0}{T_0} = \frac{1}{T_0}(\rho_0 u_{tt})$$
$$= \frac{1}{c^2} v_{tt}$$

4.4.2 $\rho_0 u_{tt} = T_0 u_{xx} + \alpha u$

(a) To understand the role of the term αu we study

$\rho_0 u_{tt} = \alpha u$ $\qquad$ $\alpha < 0$: acceleration u_{tt} is negative for positive u, hence restoring force

$\alpha > 0$: $u_{tt} > 0$ for $u > 0$ and the string is pushed away from 0.

(b) $u(x,t) = \phi(x)\, h(t)$

$$\rho_0 \phi h'' = T_0 \phi'' h + \alpha \phi h$$

$$\frac{h''}{h} = \frac{T_0}{\rho_0 \phi}\phi'' + \frac{\alpha}{\rho_0} = -\lambda$$

time equation: $h'' = -\lambda h$.

Depending on the sign of λ it has either oscillating or exponential solutions.

(c) $u(0,t) = 0 \implies \phi(0) = 0$

$u(L,t) = 0 \implies \phi(L) = 0$

$u(x,0) = 0$

$u_t(x,0) = f(x)$ comes later.

$\rho_0 = \text{const}, \quad \alpha < 0 \ \text{const}:$

$$\left.\begin{array}{l} \frac{T_0}{\rho_0}\phi'' + \frac{\alpha}{\rho_0}\phi = -\lambda\phi \\ \phi(0) = 0, \quad \phi(L) = 0 \end{array}\right\}$$

$$\phi'' = \underbrace{\frac{\rho_0}{T_0}\left(-\frac{\alpha}{\rho_0} - \lambda\right)}_{-\varkappa}\phi$$

$$\left.\begin{array}{l} \phi'' = -\varkappa\phi \\ \phi(0) = 0, \quad \phi(L) = 0 \end{array}\right\} \implies \begin{array}{l} \varkappa_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \dots \\ \phi_n = \sin\left(\frac{n\pi x}{L}\right) \end{array}$$

$$\varkappa_n = \left(\frac{n\pi}{L}\right)^2 = \frac{\rho_0}{T_0}\left(\frac{\alpha}{\rho_0} + \lambda_n\right) \implies \lambda_n = \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{\rho_0}$$

>0 since $\underline{\alpha < 0}$.

time problem : $h'' = -\lambda_n h$ $\qquad \lambda_n > 0$:

$$h(t) = \alpha \cos(\sqrt{\lambda_n}\, t) + \beta \sin(\sqrt{\lambda_n}\, t)$$

Superposition

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(\sqrt{\lambda_n}\, t) + \beta_n \sin(\sqrt{\lambda_n}\, t)\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L 0 \cdot \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

$$b_n = \frac{2}{L\sqrt{\lambda_n}} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Frequencies of vibration:

$$\sqrt{\lambda_n} = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{\rho_0}}$$

4.4.3 $\quad \rho_0 u_{tt} = T_0 u_{xx} - \beta u_t$

(a) Consider $\rho_0 u_{tt} = -\beta u_t$. For $\beta > 0$ the acceleration is opposite to the velocity, i.e. friction.

(b) ρ_0, T_0 constant.

Separation: $u(x,t) = \phi(x) h(t)$

$$\rho_0 \phi h'' = T_0 \phi'' h - \beta \phi h'$$

$$\frac{\rho_0 h'' + \beta h'}{T_0 h} = \frac{\phi''}{\phi} = -\lambda$$

: $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2$

t-problem: $\qquad \rho_0 h'' + \beta h' = -\lambda T_0 h$

$$h'' + \frac{\beta}{\rho_0} h' = -\frac{\lambda T_0}{\rho_0} h$$

auxiliary equation

$$\mu^2 + \frac{\beta}{\rho_0}\mu = -\frac{\lambda_n T_0}{\rho_0}$$

$$\mu^2 + \frac{\beta}{\rho_0}\mu + \frac{\lambda_n T_0}{\rho_0} = 0$$

$$\mu_{1/2} = -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{\rho_0^2} - 4\frac{\lambda_n T_0}{\rho_0}}$$

Assumption: $\qquad \beta^2 < \frac{4\pi^2}{L^2}\rho_0 T_0$

$\Rightarrow \qquad \frac{\beta^2}{\rho_0^2} - 4\frac{\lambda_n T_0}{\rho_0} < 0$ for all λ_n. Hence complex μ:

$$\mu_{1/2} = a \pm i b_n \qquad a = -\frac{\beta}{2\rho_0}$$

$$b_n = \frac{1}{2}\sqrt{4\frac{\lambda_n T_0}{\rho_0} - \frac{\beta^2}{\rho_0^2}}$$

general solution:

$$h(t) = \cancel{c_4}\, e^{at}\left(c_1 e^{ib_n t} + c_2 e^{-ib_n t}\right)$$

! Simpler way see page Sol 3.7

<u>Superposition</u>

$$u(x,t) = \sum_{n=1}^{\infty} e^{at}\left(A_n e^{ib_n t} + B_n e^{-ib_n t}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Initial conditions:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad A_n + B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \left(a(A_n + B_n) + (ib_n A_n - ib_n B_n)\right)$$

$$\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad a(A_n + B_n) + ib_n(A_n - B_n) = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

~~real~~ ~~complex~~ ~~real~~

~~Hence $A_n = B_n$ (the right hand side has no complex part.~~

Call $\quad \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = F_n$

$\quad \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx = G_n$

We need to solve $\quad A_n + B_n = F_n$

and $\quad a(A_n + B_n) + ib_n(A_n - B_n) = G_n$

$$\Rightarrow \quad A_n = F_n - B_n$$

$$a F_n + i b_n (F_n - 2B_n) = G_n$$

$$\Rightarrow \quad B_n = \frac{i}{2b_n}\left(G_n - aF_n - ib_n F_n \right)$$

$$= \frac{i}{2b_n}\left(G_n - aF_n \right) + \frac{F_n}{2}$$

$$A_n = F_n - B_n = \frac{F_n}{2} - \frac{i}{2b_n}\left(G_n - aF_n \right)$$

Then the solution looks like follows:

$$u(x,t) = \sum_{n=1}^{\infty} e^{at}\left[\left(\frac{F_n}{2} - \frac{i}{2b_n}\left(G_n - aF_n\right)\right)e^{ib_n t} + \left(\frac{F_n}{2} + \frac{i}{2b_n}\left(G_n - aF_n\right)\right)\right.$$

$$\left. \cdot e^{-ib_n t}\right] \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} e^{at}\left(F_n \cos(b_n t) + \frac{G_n - aF_n}{b_n} \sin(b_n t)\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$a = -\frac{\beta}{2\rho_0}, \quad b_n = \frac{1}{2}\sqrt{4\frac{\lambda_n T_0}{\rho_0} - \frac{\beta^2}{\rho_0}}$$

$$F_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$G_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

! It is simpler to write the general solution of $h(t)$ with sin and cosine:

$$h(t) = e^{at}\left(c_1 \cos(b_n t) + c_2 \sin(b_n t)\right)$$

and then do superposition etc. etc. etc.

It will give the same answer.

<u>5.3.4 a) + c)</u>

Separation: $u(x,t) = \phi(x)\,h(t)$

$$\phi h' = (k\phi'' - V_0 \phi')h$$

$$\frac{h'}{h} = k\frac{\phi''}{\phi} - V_0\frac{\phi'}{\phi} = -\lambda$$

x-problem $k\phi'' - V_0\phi' + \lambda\phi = 0$ is not of the form

$$p\phi'' + p'\phi' + q\phi + \lambda\sigma\phi = 0,$$ since we

would need $p = k$ and $p' = V_0$. That doesn't work.

c) $u_x(0,t) = 0$, $u_x(L,t) = 0 \Rightarrow \phi'(0) = 0$, $\phi'(L) = 0$.

x-problem: $\left.\begin{aligned} &k\phi' - V_0\phi' + \lambda\phi = 0 \\ &\phi'(0) = 0,\ \phi'(L) = 0 \end{aligned}\right\}$

auxiliary equation: $k\mu^2 - V_0\mu + \lambda = 0$

$$\mu^2 - \frac{V_0}{k}\mu + \frac{\lambda}{k} = 0$$

$$\mu_{1/2} = \frac{V_0}{2k} \pm \frac{1}{2}\sqrt{\underbrace{\frac{V_0^2}{k^2} - 4\frac{\lambda}{k}}}$$

make cases for $\frac{V_0^2}{k^2} - 4\frac{\lambda}{k} \gtreqless 0$

<u>Case 1</u>: $\frac{V_0^2}{k^2} - 4\frac{\lambda}{k} < 0 \iff \lambda > \frac{V_0^2}{4k}$

Then $\phi(x) = e^{\frac{V_0}{2k}x}\left(a\cos(Dx) + b\sin(Dx)\right) \qquad D = \frac{1}{2}\sqrt{4\frac{\lambda}{k} - \frac{V_0^2}{k^2}}$

$$\phi' = \frac{V_0}{2k} e^{\frac{V_0 x}{2k}}\left(a\cos(Dx) + b\sin(Dx)\right)$$

$$+ e^{\frac{V_0}{2k}x}\left(-aD\sin(Dx) + bD\cos(Dx)\right)$$

$$\phi'(0) = 0 = \frac{V_0}{2k}a + bD \Rightarrow b = -\frac{V_0}{2kD}a$$

$$\phi'(L) = 0 = \frac{V_0}{2k} e^{\frac{V_0 L}{2k}}\left(a\cos(DL) + b\sin(DL)\right)$$

$$+ e^{\frac{V_0}{2k}L}\left(-aD\sin(DL) + bD\cos(DL)\right)$$

$$\Rightarrow \underbrace{\left(\frac{V_0}{2k}a + bD\right)}_{=0}\cos(DL) = \left(aD - \frac{V_0}{2k}b\right)\sin(DL)$$

$$\Rightarrow a\left(D + \left(\frac{V_0}{2k}\right)^2\frac{1}{D}\right)\sin(DL) = 0$$

$$\Rightarrow \sin(DL) = 0 \Rightarrow DL = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\Rightarrow 4\frac{\lambda}{k} - \frac{V_0^2}{k^2} = 4(n\pi)^2$$

$$\boxed{\begin{array}{c}\lambda_n = k\left((n\pi)^2 + \frac{V_0^2}{4k^2}\right) \\ n = 1, 2, 3, \ldots\end{array}}$$

Sol 3.8

Case 2 $\lambda = \frac{V_0^2}{4k}$

$$\phi(x) = e^{\frac{V_0}{2k}x}(Ax+B)$$

$$\phi' = \frac{V_0}{2k} e^{\frac{V_0}{2k}x}(Ax+B) + e^{\frac{V_0}{2k}x} A$$

$$\phi'(0) = 0 = \frac{V_0}{2k} B + A \qquad A = -\frac{V_0}{2k} B$$

$$\phi'(L) = 0 = \frac{V_0}{2k} e^{\frac{V_0 L}{2k}}(AL+B) + e^{\frac{V_0}{2k}L} A$$

$$\Rightarrow AL = 0 \Rightarrow A = 0 \Rightarrow B = 0. \text{ no sol.}$$

Case 3: $\lambda < \frac{V_0^2}{4k}$

$$\phi(x) = C_1 e^{\mu_1 x} + C_2 e^{\mu_2 x}$$

$$\phi'(x) = C_1\mu_1 e^{\mu_1 x} + C_2\mu_2 e^{\mu_2 x}$$

$$\phi'(0) = 0: C_1\mu_1 + C_2\mu_2 = 0 \Rightarrow C_2 = -\frac{\mu_1}{\mu_2} C_1$$

$$\phi'(L) = 0: C_1\mu_1 e^{\mu_1 L} + C_2\mu_2 e^{\mu_2 L} = 0$$

$$C_1\left(\mu_1 e^{\mu_1 L} - \mu_1 e^{\mu_2 L}\right) = 0$$

$$\Rightarrow C_1 = 0 \Rightarrow C_2 = 0. \text{ no sol.}$$

We only get solutions from case 1 and these are:

$$\lambda_n = k\left((n\pi)^2 + \frac{V_0^2}{4k^2}\right) \qquad n = 1, 2, 3, \ldots$$

$$\phi_n(x) = a\, e^{\frac{V_0}{2k}x}\left(\cos(Dx) - \frac{V_0}{2kD}\sin(Dx)\right)$$

with $DL = n\pi \Rightarrow D = \frac{n\pi}{L}$, hence

$$\phi_n(x) = a\, e^{\frac{V_0}{2k}x}\left(\cos\left(\frac{n\pi x}{L}\right) - \frac{V_0 L}{2kn\pi}\sin\left(\frac{n\pi x}{L}\right)\right)$$

t-problem: $h_n(t) = c_n e^{-\lambda_n t}$

Superposition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} e^{\frac{V_0}{2k}x}\left(\cos\left(\frac{n\pi x}{L}\right) - \frac{V_0 L}{2kn\pi}\sin\left(\frac{n\pi x}{L}\right)\right)$$

Initial condition:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{generalized F-series}$$

$$a_n = \frac{\int_0^L f(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

(5.3.5.) As everybody knows by now, the Sturm-Liouville

a) eigenvalue problem $\phi'' + \lambda\phi = 0$, $\phi_x(0) = 0$, $\phi_x(L) = 0$ has eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and eigenfunctions $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ for $n = 0, 1, 2, 3, \ldots$

b) The leading eigenvalue is $\lambda_0 = 0$, $\phi_0 = 1$, and the n-th eigenvalue is actually $\lambda_{\cancel{n}\,n-1}$, $\phi_{\cancel{n}\,n-1}$, So $\phi_{\cancel{n}\,n-1} = \cos\left(\frac{\cancel{n}\,n-1}{L}\pi x\right)$ has $n-1$ zeroes.

c) From the Fourier-cosine series theory we know that $\phi_n(x)$ form an orthogonal basis of $PWS[0,L]$.

d) $$R(\phi) = \frac{\int_0^L \phi_x^2\,dx}{\int_0^L \phi^2\,dx} \geq 0.$$ If $\lambda = 0$ is an eigenvalue, then $\phi_x = 0$, hence $\phi = \text{const.}$ Which is possible.

(5.3.8) $\phi'' - x^2\phi + \lambda\phi = 0$, $\phi'(0) = 0$, $\phi'(1) = 0$ is a regular SL-problem with $p = 1$, $q = -x^2 \leq 0$, $\sigma = 1$

$[-p\phi\phi']_0^1 = 0$, hence $\lambda_1 \geq 0$.

If $\lambda = 0$ is an eigenvalue, then

$$0 = \frac{\int_0^1 (\phi')^2 + x^2\phi^2\,dx}{\int_0^1 \phi^2\,dx}$$

hence $\phi = 0$. So $\lambda = 0$ is <u>not</u> an eigenvalue.

(5.4.1)

(a) Use separation $u(x,t) = h(t)\,\phi(x)$

$$h' = -\lambda h$$

$$\left.\begin{array}{l} \frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \alpha\phi + c\rho\lambda\phi = 0 \\ \phi(0) = 0, \quad \phi(L) = 0 \end{array}\right\} (*)$$

a) $R(\phi) = \dfrac{\int_0^L K_0 \phi_x^2 - \alpha\phi^2 dx}{\int_0^L \phi^2 \rho c \, dx} \geq 0$ for $\alpha < 0$ hence $\lambda \geq 0$.

$\lambda = 0$ impossible, hence $\lambda > 0$.

b) (*) is a regular SL-problem. Hence it has an infinite sequence of eigenvalues and eigenfunctions (λ_n, ϕ_n). The h-equation is solved by $h_n(t) = c_n e^{-\lambda_n t}$ and superposition gives

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \phi_n(x).$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x), \quad c_n = \frac{\int_0^L \phi_n f c\rho \, dx}{\int_0^L \phi_n^2 c\rho \, dx}$$

c) Since all $\lambda_n > 0$ we have $e^{-\lambda_n t} \longrightarrow 0$ for $t \to \infty$. Hence $u(x,t) \longrightarrow 0$ for $t \to \infty$.

(5.5.8) (I like this one, since I recently published a paper on the fourth order derivative. If you are interested, you can download it from my webpage / publication No. 32)

(a) $uLv - vLu = uv_{xxxx} - vu_{xxxx}$

$$uv_{xxxx} = \frac{d}{dx}(uv_{xxx}) - u_x v_{xxx}$$

$$= \frac{d}{dx}(uv_{xxx}) - \frac{d}{dx}(u_x v_{xx}) + u_{xx}v_{xx}$$

$$vu_{xxxx} = \frac{d}{dx}(vu_{xxx}) - \frac{d}{dx}(v_x u_{xx}) + u_{xx}v_{xx}$$

$$\Rightarrow \quad uv_{xxxx} - vu_{xxxx} = \frac{d}{dx}\left(uv_{xxx} - vu_{xxx} + v_x u_{xx} - u_x v_{xx}\right)$$

(b) $$\int_0^1 uLv - vLu \, dx = \left[uv_{xxx} - vu_{xxx} + v_x u_{xx} - u_x v_{xx}\right]_0^1$$

$$\cancel{= -u(0)v_{xxx}(0) + v(0)u_{xxx}(0)}$$
$$\cancel{- v_x(0)u_{xx}(0) + u_x(0)v_{xx}(0)}$$
$$\cancel{+ u(1)v_{xxx}(1) + v(1)u_{xxx}(1)}$$
$$-$$

$$= u(1)v_{xxx}(1) - v(1)u_{xxx}(1) + v_x(1)u_{xx}(1) - u_x(1)v_{xx}(1)$$
$$- u(0)v_{xxx}(0) + v(0)u_{xxx}(0) - v_x(0)u_{xx}(0) + u_x(0)v_{xx}(0).$$

(c) In the above formula we obtain

$$= 0 - 0 + 0 - 0$$
$$- 0 + 0 - 0 + 0$$

(d) Another example is $\phi_{xxx}(0) = 0 = \phi_{xxx}(1)$

$$\phi_x(0) = 0 = \phi_x(1)$$

(e) Take two e-values, e-functions $(\lambda_n, \phi_n), (\lambda_m, \phi_m)$.

$$0 = \int_0^1 \phi_n L \phi_m - \phi_m L \phi_n \, dx$$

$$= \int_0^1 \phi_n(-\lambda_m e^x \phi_m) - \phi_m(-\lambda_n e^x \phi_n) \, dx$$

$$= (\lambda_n - \lambda_m) \int_0^1 \phi_n \phi_m e^x \, dx$$

If $\lambda_n \neq \lambda_m$ then $\int_0^1 \phi_n \phi_m e^x \, dx = 0$. This means ϕ_n and ϕ_m are orthogonal with weighting function e^x.

(5.5A.2) I use the trace-determinant formula:

$$\lambda_{1/2} = \frac{trA}{2} \pm \frac{1}{2}\sqrt{trA^2 - 4\det A}$$

(a) $trA = 2 \quad \det A = 1$

$$\lambda_{1/2} = 1 \pm \frac{1}{2}\sqrt{4-4} = 1$$

e-vector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$: $\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \implies \xi_1 = 0$,

hence $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a unique eigenvector.

(b) $trA = 1$, $\det A = 1$, well A is the identity matrix, hence ~~every~~ each vector is eigenvector!

e.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 17 \\ 5 \end{pmatrix}$ or $\begin{pmatrix} 10000001 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 \\ 171 \end{pmatrix}$

etc. . . -

(5.5A.3) $\mathrm{tr}\, A = 9 \quad \det A = 18 - 4 = 14$

$$\lambda_{1/2} = \frac{9}{2} \pm \frac{1}{2}\sqrt{81-56} = \frac{9}{2} \pm \frac{1}{2}\sqrt{25} = \frac{9}{2} \pm \frac{5}{2}$$

$$= \begin{cases} 7 \\ 2 \end{cases}$$

eigenvectors: $\lambda_1 = 7$: $\begin{pmatrix} 1 & -4 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \quad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda_2 = 2 \quad \begin{pmatrix} -4 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(a) $(4,1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 4 - 1 = 3 \quad \neq 0$

(b) $(4,1) \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{4} \cdot 4 - 1 = 0$ how exciting!

(5.5A.4)

(a) $\mathrm{tr}\, A = 9 \quad \det A = 14$

$$\lambda_{1/2} = \frac{9}{2} \pm \frac{1}{2}\sqrt{25} = \begin{cases} 7 \\ 2 \end{cases}$$

$\lambda_1 = 7$: $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$: $\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

general solution: $v(t) = c_1 e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

initial condition: $v(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow \quad 1 = 2c_1 + c_2 \quad \Rightarrow \quad c_2 = 1 - 2c_1$

$\quad\;\; 2 = c_1 - 2c_2 \quad \Rightarrow \quad 2 = c_1 - 2 + 4c_1$

$\quad\quad \Rightarrow \quad c_1 = \frac{4}{5}, \quad c_2 = -\frac{3}{5}$

$$v(t) = \frac{4}{5}e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{3}{5}e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(b) $\operatorname{tr} A = 3 \quad \det A = -4 - 4 = -8$

$$\lambda_{1/2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 + 32} = \frac{3}{2} \pm \frac{1}{2}\sqrt{41}$$

$$= a \pm b \qquad a = \frac{3}{2}, \quad b = \frac{\sqrt{41}}{2}$$

$$\begin{pmatrix} a \pm b + 1 & -2 \\ -2 & a \pm b - 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \qquad \cancel{\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a \pm b + 1 \end{pmatrix}}$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ a \pm b + 1 \end{pmatrix}$$

General solution

$$v(t) = c_1 e^{(a+b)t} \begin{pmatrix} 2 \\ a+b+1 \end{pmatrix} + c_2 e^{(a-b)t} \begin{pmatrix} 2 \\ a-b+1 \end{pmatrix}$$

$$v(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ a+b+1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ a-b+1 \end{pmatrix}$$

$$2 = 2c_1 + 2c_2 \Rightarrow c_1 = 1 - c_2$$

$$3 = (1 - c_2)(a+b+1) + c_2(a-b+1)$$

$$= a + b + 1 - 2c_2 b$$

$$2 = \frac{3}{2} + \frac{\sqrt{41}}{2} - c_2\sqrt{41}$$

$$\frac{1}{2} - \frac{\sqrt{41}}{2} = -\sqrt{41}\, c_2 \Rightarrow c_2 = \frac{\sqrt{41} - 1}{2\sqrt{41}}$$

$$c_1 = 1 - c_2 = \frac{2\sqrt{41} - \sqrt{41} + 1}{2\sqrt{41}} = \frac{\sqrt{41} + 1}{2\sqrt{41}}$$

$$v(t) = \frac{1}{2\sqrt{41}}\left((\sqrt{41} + 1) e^{\left(\frac{3}{2} + \frac{\sqrt{41}}{2}\right)t} \begin{pmatrix} 2 \\ \frac{5}{2} + \frac{\sqrt{41}}{2} \end{pmatrix} + (\sqrt{41} - 1) e^{\left(\frac{3}{2} - \frac{\sqrt{41}}{2}\right)t} \begin{pmatrix} 2 \\ \frac{5}{2} - \frac{\sqrt{41}}{2} \end{pmatrix} \right)$$

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