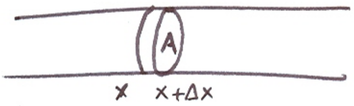
MATH 300

Assignment #1 Solutions:

1.2.2 Constant thermal property and no source

(a)



Heat energy for the slice $= e(x,t)A\Delta x$, Heat flux $= \phi(x,t)$

rate of change of energy in slice = flow of energy throughout the slice

$$\frac{\partial}{\partial t} e(x,t) A\Delta x \approx \left[\phi(x,t)A - \phi(x+\Delta x, t)\right] A$$

$$\frac{\partial}{\partial t} e(x,t) \approx \frac{\phi(x,t) - \phi(x+\Delta x, t)}{\Delta x}$$

as $\Delta x \to 0$

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}} \quad \text{I}$$

energy per unit mass $= c\,u(x,t)$

total mass $= \rho c u(x,t)$

total thermal energy $= \rho c u(x,t) A \Delta x$

$$e(x,t)A\Delta x = \rho c u(x,t) A\Delta x$$

$\Rightarrow e(x,t) = \rho c\, u(x,t)$ plug into I

$$\therefore \boxed{\rho c \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial t}} \quad \text{II}$$

Fourier's Law of heat conductivity : $\phi = -K_0 \dfrac{\partial u}{\partial x}$ plug into II

$$\therefore \rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} \Longrightarrow \boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$

(b) total energy $= \displaystyle\int_a^b e(x,t)\, A\, dx$ = sum of infinitesimal slices

$$\frac{d}{dt}\int_a^b e\, dx = \phi(a,t) - \phi(b,t)$$

$$\int_a^b \frac{\partial e}{\partial t}\, dx = -\int_a^b \frac{\partial \phi}{\partial x}\, dx$$

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x}\right) dx = 0 \Longrightarrow \frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} = 0$$

thermal energy $= \rho c\, u(x,t)\, A\, \Delta x$

$$\Longrightarrow e(x,t) = \rho c\, u(x,t)$$

Fourier's Law of heat conductiviy $\phi = -K_0 \dfrac{\partial u}{\partial x}$

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} = 0 \Longrightarrow \rho c \frac{\partial u}{\partial t} - K_0 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Longrightarrow \frac{\partial u}{\partial t} = \frac{K_0}{\rho c} \frac{\partial^2 u}{\partial x^2} \Longrightarrow \boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$

2.3.2 (a)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

(a) $\phi(0) = 0 \qquad \phi(\pi) = 0$

To solve $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ we consider the auxilary equation

$$r^2 + \lambda = 0 \implies r^2 = -\lambda$$

If $\lambda = 0 \implies \phi(x) = c_1 x + c_2$

$\phi(0) = 0 \implies c_2 = 0$

$\phi(\pi) = 0 \implies c_1 = 0$ trivial solution

If $\lambda < 0$ $\quad r^2 = -\lambda$ has two real roots $\pm\sqrt{-\lambda}$

$$\phi(x) = c_1 e^{x\sqrt{-\lambda}} + c_2 e^{-x\sqrt{-\lambda}}$$

$\phi(0) = 0 \implies c_1 + c_2 = 0 \implies c_1 = -c_2$

$\phi(\pi) = 0 \implies c_1 e^{\pi\sqrt{-\lambda}} + \frac{c_2}{e^{\pi\sqrt{-\lambda}}} = 0 \implies c_2\left[\frac{-e^{2\pi\sqrt{-\lambda}} + 1}{e^{\pi\sqrt{-\lambda}}}\right] = 0$

$c_2 = 0$ or $-e^{-2\pi\sqrt{-\lambda}} + 1 = 0 \implies \lambda = 0$ not possible

$\implies c_2 = 0 \implies c_1 = 0$ trivial case

If $\lambda > 0$ $\quad r^2 = -\lambda$ has two imaginary solutions $r = \pm i\sqrt{\lambda}$

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$\phi(0) = 0 \implies c_1 = 0$

$\phi(\pi) = 0 \implies c_2 \sin(\sqrt{\lambda}\pi) = 0 \implies \sqrt{\lambda}\pi = n\pi \quad \lambda = n^2$

$\boxed{\phi_n(x) = C_n \sin(nx)}$ with $\boxed{\lambda_n = n^2}$ and $n = 1, 2, 3, \ldots$

.3.2 (e) $\frac{d\phi}{dx}(0) = 0$ $\quad \phi(L) = 0$

$\frac{d^2\phi}{dx} + \lambda\phi = 0$ the auxilary equ. is $r^2 + \lambda = 0$

$r^2 = -\lambda$

$\underline{\lambda = 0}$ $\Rightarrow$ $\phi(x) = c_1 x + c_2$ $\quad \frac{d\phi}{dx} = c_1$

$\frac{d\phi}{dx}(0) = 0 \Rightarrow c_1 = 0$

$\phi(L) = 0 \Rightarrow c_1 L + c_2 = 0 \Rightarrow c_2 = 0$ $\quad$ trivial

If $\lambda < 0$ $\quad \phi(x) = c_1 e^{x\sqrt{-\lambda}} + c_2 e^{-x\sqrt{-\lambda}}$ $\quad \frac{d\phi}{dx} = +\sqrt{-\lambda}\left(c_1 e^{x\sqrt{-\lambda}} - c_2 e^{-x\sqrt{-\lambda}}\right)$

$\frac{d\phi}{dx}(0) = 0 \Rightarrow c_1 - c_2 = 0 \Rightarrow \boxed{c_1 = c_2}$

$\phi(L) = 0 \Rightarrow c_1 e^{L\sqrt{-\lambda}} + c_1 e^{-L\sqrt{-\lambda}} = 0 \Rightarrow c_1\left(\frac{e^{2L\sqrt{-\lambda}} + 1}{e^{L\sqrt{-\lambda}}}\right) = 0$

$\Rightarrow \boxed{c_1 = 0}$ trivial $\quad$ notice $e^{2L\sqrt{-\lambda}} + 1 \neq 0$ since λ is real

$\left[\text{ie. } e^{2L\sqrt{-\lambda}} \neq -1\right]$

If $\lambda > 0$ $\quad$ Two imaginary roots

$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$ $\quad \frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$\frac{d\phi}{dx}(0) = 0 \Rightarrow c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0$

$\phi(x) = c_1\cos(\sqrt{\lambda}x)$, $\phi(L) = 0 \Rightarrow c_1\cos(\sqrt{\lambda}L) = 0$

$\Rightarrow \sqrt{\lambda}L = \frac{(2n-1)\pi}{2} \Rightarrow \lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$ for $n = 1, 2, 3, \ldots$

$$\boxed{\begin{aligned}\phi_n(x) &= c_1\cos\left(\frac{(2n-1)\pi}{2L}x\right)\\ \lambda_n &= \left(\frac{(2n-1)\pi}{2L}\right)^2\end{aligned}}$$

2.3.3 (b) $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = 0 \quad u(L,t) = 0$

with $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

let $u(x,t) = \phi(x)\,G(t)$ and plug into the heat equation and divide by $K\,G(t)\,\phi(x)$

$$\Rightarrow \quad \frac{1}{G}\frac{dG}{dt} = K\frac{d^2\phi}{dx^2}\frac{1}{\phi} = -\lambda$$

$$\Rightarrow \begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \qquad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0 \\[2ex] \dfrac{dG}{dt} = -\lambda K G \end{cases}$$

We saw that $\lambda = 0$ and $\lambda < 0$ yeild to trivial solutions

then $\lambda > 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \phi(x) = \sin\frac{n\pi x}{L}$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-K\left(\frac{n\pi}{L}\right)^2 t}$$

use the initial condition and the principle of superposition we get:

$$\boxed{u(x,t) = 3\sin\frac{\pi x}{L}\, e^{-K\left(\frac{\pi}{L}\right)^2 t} - \sin\frac{3\pi x}{L}\, e^{-K\left(\frac{3\pi}{L}\right)^2 t}}$$

2.3.5 $\quad I = \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \frac{1}{2}\left[\cos\left(\frac{n\pi x - m\pi x}{L}\right) - \cos\left(\frac{n\pi x + m\pi x}{L}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right]$$

Case 1 $\quad m = n \neq 0$

$$\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \frac{1}{2}\left[1 - \cos\frac{2n\pi x}{L}\right]$$

$$I = \frac{1}{2}\int_0^L \left(1 - \cos\frac{2n\pi x}{L}\right)dx = \frac{1}{2}\int_0^L dx - \frac{1}{2}\int_0^L \cos\frac{2n\pi x}{L}\,dx$$

$$= \frac{x}{2}\Big|_{x=0}^{L} - \frac{L}{2n\pi}\left[\sin\left(\frac{2n\pi x}{L}\right)\right]_{x=0}^{L}$$

$$= \frac{L}{2} - \frac{L}{2\pi n}\left(\cancelto{0}{\sin(2n\pi)} - \cancelto{0}{\sin(0)}\right)$$

$$= \boxed{\frac{L}{2}}$$

Case 2: $\quad m \neq n$

$$I = \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right)dx - \frac{1}{2}\int_0^L \cos\left(\frac{(m+n)\pi x}{L}\right)dx$$

$$= \frac{L\sin\left(\frac{(n-m)\pi x}{L}\right)}{2(n-m)\pi}\Bigg|_0^L - \frac{L\sin((m+n)\pi x)}{2((m+n)\pi)}\Bigg|_0^L$$

$$= \frac{L}{2(n-m)\pi}\left(\cancelto{0}{\sin((n-m)\pi)} - \cancelto{0}{\sin(0)}\right) - \frac{L}{2(n+m)\pi}\left[\cancelto{0}{\sin(m+n)\pi} - \cancelto{0}{\sin(0)}\right] = \boxed{0}$$

$$\Rightarrow \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m\neq n \quad \text{or} \quad m=0 \text{ and } n=0 \\ \frac{L}{2} & m=n\neq 0 \end{cases}$$

2.3.7 $\quad \dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2} \qquad \dfrac{\partial u}{\partial x}(0,t)=0 \quad \dfrac{\partial u}{\partial x}(L,t)=0$

$u(x,0)=f(x)$

(a) The heat flow in one dimensional rod, with all constant thermal properties, and no source with no heat flow at the boundaries of the rod.

(b) $u(x,t)=G(t)\phi(x) \xrightarrow{\text{plug into equ.}} \begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda\phi & \dfrac{d\phi}{dx}(0)=0 \\ & \dfrac{d\phi}{dx}(L)=0 \\ \text{and} & \\ G(t)= ce^{-\lambda K t} & \end{cases}$

To show that there is no exponentialy ~~growth~~ growth in time we must show that λ can not be negative.

Let assume $\lambda<0$

$\dfrac{d^2\phi}{dx^2} = -\lambda\phi \Rightarrow$ auxilary equation $r^2=-\lambda$

$\phi(x) = c_1e^{x\sqrt{-\lambda}} + c_2e^{-x\sqrt{-\lambda}}, \quad \phi'(x) = -\sqrt{\lambda}\left[c_1e^{x\sqrt{-\lambda}} - c_2e^{-x\sqrt{-\lambda}}\right]$

$\phi'(0)=0 \Rightarrow \boxed{c_1=c_2} \qquad \phi'(L)=0 \Rightarrow$

$\longrightarrow$

$$\phi'(L)=0 \Rightarrow c_1\left[e^{L\sqrt{-\lambda}}-\frac{1}{e^{L\sqrt{-\lambda}}}\right]=0 \Rightarrow c_1\left[\frac{e^{2L\sqrt{-\lambda}}-1}{e^{L\sqrt{-\lambda}}}\right]=0$$

$$c_1=0 \quad \text{or} \quad \frac{e^{2L\sqrt{-\lambda}}-1}{e^{L\sqrt{-\lambda}}}=0$$

$$e^{2L\sqrt{-\lambda}}-1=0 \Rightarrow 2L\sqrt{-\lambda}=0 \Rightarrow \lambda=0 \quad \text{but } \lambda<0$$

$$\Rightarrow c_1=0 \Rightarrow c_2=0 \Rightarrow \text{trivial solution}$$

$\Rightarrow$ no separated solutions which grow exponentially in time.

$$\lambda=0 \Rightarrow \phi(x)=c_1x+c_2 \quad \text{apply B.C} \Rightarrow \phi(x)=c$$

$$\lambda>0 \Rightarrow \phi(x)=c_1\cos(\sqrt{-\lambda}x)+c_2\sin(\sqrt{-\lambda}x)$$

Apply conditions: $\Rightarrow$ $\boxed{\begin{array}{c}\phi_n(x)=C_n\cos\frac{n\pi x}{L}\\ \lambda_n=\left(\frac{n\pi}{L}\right)^2\end{array}}$

$$\Rightarrow u(x,t)=A_0+\sum_{n=1}^{\infty}A_n e^{-\lambda_n kt}\cos\frac{n\pi x}{L}, \quad \lambda_n=\left(\frac{n\pi}{L}\right)^2$$

$n=1,2,\dots$

(c) $$u(x,0)=A_0+\sum_{n=1}^{\infty}A_n e^{0}\cos\frac{n\pi x}{L}$$

$$=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right) \Rightarrow \text{If } f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)$$

then $u(x,0)=f(x)$ is satisfied

(d) $$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ L & m = n = 0 \end{cases}$$

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$$

Multiply both sides of $f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$ by $\cos\frac{m\pi x}{L}$

and integrate over the domain:

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = \sum_{n=0}^{\infty}\int_0^L A_n \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = \begin{cases} \int_0^L A_0\,dx = L A_0 & \text{if } m = n = 0 \\ \frac{L}{2} & \text{otherwise} \end{cases}$$

$$\Longrightarrow \begin{cases} A_0 = \frac{1}{L}\int_0^L f(x)\,dx \\ A_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx & m \geq 1 \end{cases}$$

(e) as $t \to \infty$

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left(A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t} \cos\frac{n\pi x}{L}\right)$$

$$= A_0 + \lim_{t\to\infty}\left(\sum_{n=1}^{\infty} A_n e^{-\lambda_n k t}\cos\left(\frac{n\pi x}{L}\right)\right)$$

$$= A_0 + 0$$

$$= A_0$$

$$= \frac{1}{L}\int_0^L f(x)dx \quad \text{by part (d)}$$

2.4.4

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \frac{d\phi}{dx}(0) = 0 \qquad \frac{d\phi}{dx}(L) = 0$$

The auxilary equation is $r^2 = -\lambda$

If $\lambda < 0 \Rightarrow r^2 = -\lambda$ has two real roots $\pm\sqrt{-\lambda}$

$\phi(x) = c_1 e^{x\sqrt{-\lambda}} + c_2 e^{-x\sqrt{-\lambda}}$ let $\sqrt{-\lambda} = s$

notice that $s \neq 0$ since $\lambda < 0$

$$\phi(x) = c_1 e^{xs} + c_2 e^{-xs}$$

$$\frac{d\phi}{dx}(x) = sc_1 e^{xs} - sc_2 e^{-xs} = s(c_1 e^{xs} - c_2 e^{-xs})$$

$$\frac{d\phi}{dx}(0) = 0 \Rightarrow s(c_1 - c_2) = 0$$

$$\frac{d\phi}{dx}(L)=0 \Rightarrow sc_1(e^{Ls}-e^{-Ls}) = sc_1\left(e^{Ls}-\frac{1}{e^{Ls}}\right)=0$$

$$\Rightarrow sc_1\left(\frac{e^{2Ls}-1}{e^{Ls}}\right)=0 \Rightarrow c_1=0$$

since $s\neq 0$ and $e^{2Ls}-1\neq 0$

$\Rightarrow c_1=0 \Rightarrow c_2=0$

we will get the trivial solution.

$\Rightarrow$ no negative eigenvalues.

2.4.7 $\nabla^2 u = \frac{1}{r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right)=0$ $u(a,\theta)=f(\theta)$

$u(x,y)$ Bounded Boundary

$0\leq x\leq L$ $0\leq y\leq H$ $\Rightarrow$ polar $\begin{cases} -\pi\leq\theta\leq\pi \\ 0\leq r\leq a\end{cases}$

$u(r,-\pi)=u(r,\pi)$

$\frac{\partial u}{\partial\theta}(r,-\pi)=\frac{\partial u}{\partial\theta}(r,\pi)$ periodic B.C. $|u(0,\theta)|<\infty$

$u(r,\theta)=\phi(\theta)G(r)$

plug into $\nabla^2 u = \ldots$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)\phi(\theta)+\frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2}=0$$

divide by $\frac{G\phi}{r^2}$ we get $\longrightarrow$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \frac{-1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda \qquad \Rightarrow$$

① $$\frac{d^2\phi}{d\theta^2} = -\lambda\phi \qquad \phi(-\pi) = \phi(\pi)$$
$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

We have seen that the solution is only non trivial for $\lambda > 0$ where $\lambda = n^2$ $\boxed{\phi(\theta) = C_1 \sin n\theta + C_2 \cos n\theta}$

② $$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda = n^2$$

$$\Rightarrow r^2 G''(r) + rG'(r) - n^2 G = 0 \quad \text{and} \quad |G(0)| < \infty$$

Let $G = r^p$ and plug into the above equation then

$$G = c_1 r^n + c_2 r^{-n} \quad \text{and} \quad |G(0)| < \infty$$

$$\Rightarrow \boxed{G = c_1 r^n}$$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$$

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta \quad \text{since } f(\theta) = u(a,\theta)$$

then $\boxed{A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta}$

$\boxed{A_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta}$ and $\boxed{B_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta}$