(8.4) The 2-D Navier-Stokes equation

$S(t)$ solution semigroup of the 2-D N-S-eq:

$$\frac{du}{dt} + \nu A u + B(u,u) = f \quad \text{on a periodic domain } Q.$$

$\mathcal{A}$ compact global attractor in $H = \{ u \in \mathbb{L}^2(Q) : \nabla u = 0 \}$,

$V = \{ u \in \mathbb{H}^1_p(Q) ;\ \nabla \cdot u = 0 \}$.

<u>Theorem 1</u> (i) $S(t)$ is uniformly differentiable on $\mathcal{A}$

(ii) The linearization $\Lambda(t; u_0)$, $u_0 \in \mathcal{A}$ is solution semigroup of the linearized equation

$$\frac{du}{dt} + \nu A u + B(u, \mathcal{U}) + B(\mathcal{U}, u) = 0$$

(iii) Λ is compact.

<u>Proof</u>: As before $u(t), v(t)$, $\mathcal{U}(t)$, $\mathcal{U}(0) = v_0 - u_0$,

$\theta = v - u - \mathcal{U}$ and we find

$$\frac{d\theta}{dt} + \nu A\theta + B(u,\theta) + B(\theta, u) + B(v-u, v-u) = 0$$

We introduce $w = u - v$, multiply the above equation by θ and integrate $(b(u,\theta,\theta) = 0)$

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 + \nu\|\theta\|^2 = -b(\theta, u, \theta) - b(w, w, \theta)$$

$$\overset{2\text{-D}}{\leq} k\,|\theta|\,\|\theta\|\,\underbrace{\|u\|}_{\leq \rho_V \text{ on the attractor}} + k\,|w|\,\|w\|\,\|\theta\|$$

$$\overset{\text{Young}}{\leq} \frac{k}{\nu}|\theta|^2 + \frac{\nu}{4}\|\theta\|^2 + \frac{k}{\nu}|w|^2\|w\|^2 + \frac{\nu}{4}\|\theta\|^2$$

$$\frac{d}{dt}|\theta|^2 + \nu\|\theta\|^2 \leq C|\theta|^2 + C|w|^2\|w\|^2$$

We disregard the $\|\theta\|^2$-term. We know that $\theta(0)=0$ and with Gronwalls lemma we get:

$$|\theta(t)|^2 \le k \int_0^t |w(s)|^2 \|w(s)\|^2 ds \qquad (*)$$

From the proof of uniqueness of the N-S-eq we get:

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 \le \frac{k^2}{\nu}\|u\|^2|w|^2 \qquad (**)$$

Disregard $\|w\|^2 \ge 0$ and integration:

$$|w(t)|^2 \le \exp\left(\int_0^t \frac{k^2}{\nu} \underbrace{\|u(s)\|^2}_{\le \rho_\nu} ds\right)|w(0)|^2$$

$$\le e^{Kt}|w_0|^2$$

Now we multiply $(**)$ by $|w|^2$ and integrate.

Note that

$$\frac{d}{dt}|w|^4 = \frac{d}{dt}\left(\int |w|^2 dx\right)^2 = 2\int |w|^2 dx \cdot \frac{d}{dt}\int |w|^2 dx$$

$$= 2|w|^2 \frac{d}{dt}\|w\|^2$$

hence

$$|w(t)|^4 - |w_0|^4 + \nu\int_0^t |w|^2\|w\|^2 ds \le \frac{k^2}{\nu}\int_0^t \underbrace{\|u\|^2}_{\le \rho_\nu} \underbrace{|w|^4}_{\le e^{2Kt}|w_0|^4} ds$$

$$\nu\int_0^t |w|^2\|w\|^2 ds \le \frac{k^2}{\nu}\rho_\nu^2 \int_0^t e^{2Kt}|w_0|^4 dt + |w_0|^4$$

$$\le C e^{2Kt}|w_0|^4 .$$

Then from $(*)$:

$$|\theta(t)|^2 \le C\,|u_0 - v_0|^4 \quad \text{and} \quad |\theta(t)| \le C|u_0 - v_0|^2$$

Finally

$$\frac{|v-u-U|}{|v_0-u_0|} \le c\,|v_0-u_0| \longrightarrow 0 \quad \text{as} \quad u_0 \to v_0 .$$

(iii) $\Lambda(t;u_0)$ is compact because there is an absorbing set in $V \subset\subset H$.

Theorem 2: There is a constant $\alpha > 0$ such that

$$d_f(A) \le \alpha \left(\frac{\varrho_v}{\nu}\right)^2$$

Proof: Linearized Operator

$$L(t;u_0) = \nu A - B(\cdot,u) - B(u,\cdot)$$

$$\langle L(t;u_0)P_n\rangle = \left\langle \sum_{j=1}^{n} (\phi_j,\ L(t;u_0)\,\phi_j) \right\rangle$$

$$= -\left\langle \sum_j (-\nu A\phi_j,\ \phi_j) \right\rangle - \left\langle \sum_j b(\phi_j,u,\phi_j) \right\rangle$$

$$\overset{2\text{-}D}{\le} -\nu \sum_j \langle \|\phi_j\|^2\rangle + k\left\langle \sum_j \|u\|\ \|\phi_j\|\ \underbrace{|\phi_j|}_{=1} \right\rangle$$

$$\le -\nu \sum_j \langle \|\phi_j\|^2\rangle + k \sum_j \langle \|u\|\,\|\phi_j\|\rangle$$

$$\overset{\text{Young}}{\le} -\nu \sum_j \langle \|\phi_j\|^2\rangle + \sum_j \left\langle \frac{\nu}{2}\|\phi_j\|^2 + \frac{k^2}{2\nu}\|u\|^2 \right\rangle$$

$$= -\frac{\nu}{2}\sum_j \langle \|\phi_j\|^2\rangle + \frac{k^2}{2\nu}\sum_j \langle \|u\|^2\rangle$$

$$= -\frac{\nu}{2}\langle \operatorname{tr}(-\Delta P_n)\rangle + \frac{k^2 n}{2\nu}\langle \|u\|^2\rangle$$

$$\Rightarrow \langle P_n L(t; u_0) \rangle \le -\frac{c\nu}{2} n^2 + \frac{h^2 n}{2\nu} \langle \|u\|^2 \rangle$$

which is negative for each

$$n > \alpha \left(\frac{\langle \|u\|^2 \rangle}{\nu^2} \right) \qquad \alpha = \frac{h^2}{c}$$

Since $\|u\|^2 \le \rho_\nu$ on $\mathcal{A}$ we find decreasing volumes for

$$n > \alpha \frac{\rho_\nu}{\nu^2}$$

$\square$