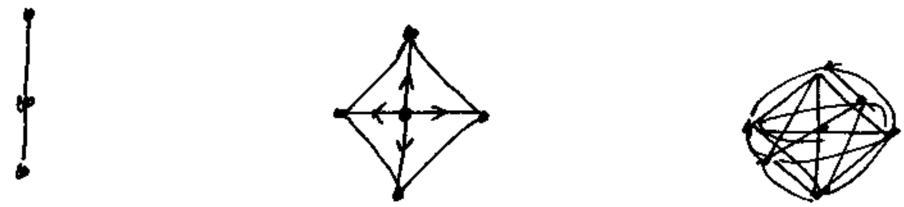
(8.3) Reaction - Diffusion Equations

Chaffee - Infante equation: $u_t = u_{xx} + \lambda(u - u^3)$



$1 < \lambda < 4$ $\quad$ $4 < \lambda < 9$ $\quad$ $9 < \lambda < 16$

$d_f(A) \geq 1$ $\quad$ $d_f(A) \geq 2$ $\quad$ $d_f(A) \geq 3$

guess: If $n^2 < \lambda < (n+1)^2$, then $d_f(A) \sim \lambda^{1/2}$

We assume that $S(t)$ is the solution semigroup of the reaction diffusion equation $\frac{du}{dt} = \Delta u + \lambda f(u)$, as found in chapter 3. f is supposed to satisfy the assumptions of (3.2) and $f \in C^2$. Let A be the compact global attractor.

Proposition 1:

(i) $S(t)$ is uniformly differentiable on A

(ii) The linearization $\Lambda(t; u_0)$ is the solution semigroup of the linearized equation

$$\frac{dU}{dt} = \Delta U + \lambda f'(u(t))U, \quad U(0) = \xi$$

(iii) $\Lambda(t; u_0)$ is compact for all $t > 0$.

<u>Proof</u>: We use the same solution theory for the linearized equation as we used for the nonlinear equation. Then $\Lambda(t;u_0)$ exists for each $t>0$ and

$$\|\Lambda(t;u)\|_{op} < \infty.$$

We need to show that

$$\sup_{\substack{v_0,u_0\in\mathcal{B}\\ 0<|u_0-v_0|<\varepsilon}} \frac{|S(t)v_0 - S(t)u_0 - \Lambda(t;u_0)(v_0-u_0)|}{|v_0-u_0|} \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

Denote $u(t)=S(t)u_0$, $v(t)=S(t)v_0$, $\mathcal{U}(t)=\Lambda(t;u_0)(v_0-u_0)$

and $\quad \theta(t) := v(t)-u(t)-\mathcal{U}(t).$

Then the error $\theta(t)$ satisfies

$$\frac{d}{dt}\theta = \Delta\theta + \lambda f'(u)\theta + \lambda\Big(\underbrace{f(v)-f(u)-f'(u)(v-u)}_{=:g(u,v).}\Big) \qquad (1)$$

We write $g(x)=g(u(x,t),v(x,t))$ and use the mean value theorem for $f\in C^2$: (or Taylor expansion)

$$|g(x)| \le \tfrac{1}{2}\,|f''(\sigma)|\,|v(x)-u(x)|^2 \text{ for some}$$

σ between $v(x)$ and $u(x)$. Since $u,v\in\mathcal{B}$ they are uniformly bounded in L^∞. Hence $f''(\sigma)$ is bounded:

$$|g(x)| \le c\,|v(x)-u(x)|^2$$

for some constant $c>0$.

Now we choose p such that $H^1 \subset L^p$ and let q denote the conjugate index. We now write $g(t) = g(u(x,t), v(x,t))$:

$$\|g(t)\|_q^q \leq C \int_\Omega |u-v|^{2q}\,dx$$

$$\leq C \int_\Omega |u-v|^{2q-2+\varepsilon}\,|u-v|^{2-\varepsilon}\,dx$$

Hölder

$$\leq C \,\big\|\, |u-v|^{2q-2+\varepsilon} \big\|_{\frac{2}{\varepsilon}} \,\big\| |u-v|^{2-\varepsilon} \big\|_{\frac{2}{2-\varepsilon}}$$

indeed: $\frac{\varepsilon}{2} + \frac{2-\varepsilon}{2} = 1$

$$= C\,\|u-v\|_{\frac{2}{\varepsilon}(2q-2+\varepsilon)}^{2q-2+\varepsilon} \quad \|u-v\|_2^{2-\varepsilon}$$

$$\leq \tilde{C}\,|u-v|^{2-\varepsilon}$$

Hence $\|g(t)\|_q \leq \tilde{C}\,|u-v|^{\frac{2-\varepsilon}{q}}$

Now we choose $\varepsilon = 2 - q(1+\delta)$ for some $\delta \in \left(0, \frac{2-q}{q}\right)$

We get $\|g(t)\|_q \leq C\,|u-v|^{1+\delta}$ (2)

From the proof of uniqueness of weak solutions we know that $|u-v| \leq e^{lt}\,|u_0 - v_0|$ $\quad (\,|f'| \leq l\,)$

hence (2) reads

$$\|g(t)\|_q \le c e^{(1+\delta)\ell t} |u_0 - v_0|^{1+\delta}$$

To estimate θ we multiply (1) by θ and integrate

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 + \|\theta\|^2 \overset{\text{Hölder}}{\le} \lambda\ell|\theta|^2 + \lambda\|g(t)\|_q \|\theta\|_p$$

$$\overset{\text{Sobolev embedding}}{\le} \lambda\ell|\theta|^2 + c\lambda e^{(1+\delta)\ell t}|u_0 - v_0|^{1+\delta}\|\theta\|_{H^1}$$

$$\overset{\text{Young}}{\le} \lambda\ell|\theta|^2 + c\lambda^2 e^{2(1+\delta)\ell t}|u_0 - v_0|^{2(1+\delta)} + \frac{1}{2}\|\theta\|^2$$

We neglect the $\|\theta\|^2$-terms:

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 \le \lambda\ell|\theta|^2 + c\lambda^2 e^{2(1+\delta)\ell t}|u_0 - v_0|^{2(1+\delta)}.$$

Gronwall:

$$|\theta(t)|^2 \le k(t)\,|u_0 - v_0|^{2(1+\delta)}$$

which gives

$$\frac{|S(t)v_0 - S(t)u_0 - \Lambda(t;u_0)(v_0 - u_0)|}{|u_0 - v_0|} \le k(t)|u_0 - v_0|^{\delta}$$

$$\longrightarrow 0 \quad \text{as } |u_0 - v_0| \to 0.$$

Compactness of $\Lambda(t;u_0)$ follows from the same estimates as used for strong solutions (multiply by Au and integrate). For each bounded set X, $\Lambda(t;u_0)X$ is bounded in $H_0^1 \subset\subset L^2$. Hence $\Lambda(t;u_0)$ is compact.

<u>Theorem 2</u>: The dimension of the global attractor $\mathcal{A}$ of the reaction-diffusion equation in $\mathbb{R}^m$

$$\frac{du}{dt} = \Delta u + \lambda f(u), \quad u\big|_{\partial\Omega} = 0$$

is bounded by

$$d_f(\mathcal{A}) \leq \left(\frac{\ell}{C}\right)^{m/2} \lambda^{m/2},$$

where the constant c is given by Lemma 3 in (8.2)

$c = \tilde{c}\,\frac{1}{1+\frac{2}{m}}$, $\tilde{c}\, j^{2/m} \leq \lambda_j$, the e-values of $-\Delta$ on $\mathcal{D}(\mathcal{A})$. $u|_{\partial\Omega} = 0$.

<u>Proof</u>: We need to estimate $\langle \mathrm{tr}\,(L(t;u_0)\,P^{(n)}(t))\rangle$ with

$$L(t;u_0) = \Delta + \lambda f'(u(t))$$

$$\begin{aligned}
\langle \mathrm{tr}\,(L(t;u_0)\,P^{(n)}(t))\rangle &= -\langle \mathrm{tr}\,(-\Delta P^{(n)})\rangle + \lambda \left\langle \sum_{j=1}^{n} (\phi_j, f'(u)\phi_j)\right\rangle \\
&\leq -c\,n^{\frac{m+2}{m}} + \lambda \ell \left\langle \sum_{j=1}^{n} |\phi_j|^2 \right\rangle \\
&= -c\,n^{\frac{m+2}{m}} + n\lambda\ell
\end{aligned}$$

Hence $\mathrm{TR}_n(\mathcal{A})$ is certainly negative once

$$c\,n^{\frac{m+2}{m}} > n\lambda\ell$$

$$n^{\frac{m+2}{m}-1} > \frac{\lambda\ell}{c}$$

$$n > \left(\frac{\ell\lambda}{c}\right)^{\frac{m}{2}} = \left(\frac{\ell}{c}\right)^{\frac{m}{2}} \lambda^{\frac{m}{2}}$$

Application of Hurd's theorem shows that

$$d_f(A) \le \left(\frac{\ell}{c}\right)^{\frac{m}{2}} \lambda^{\frac{m}{2}} .$$

$\square$

Example: Chaffee-Infante equation:

The e-values of $-\Delta$ on $[0,\pi]$ with homogeneous Dirichlet boundary conditions $u(0)=0$, $u(\pi)=0$ are

$C = \frac{\tilde{C}}{3}$ $\qquad \lambda_j = j^2 = j^{2/1} \quad (m=1). \quad \Rightarrow \quad c = \frac{1}{3}$

$\tilde{C} = 1$

The nonlinearity $f(u) = u - u^3$ has derivative $f'(u) = 1 - 3u$. We know that the attractor is contained in the set $\{0 \le u \le 1\}$.

$$\max_{0 \le u \le 1} |f'(u)| = 2 = \ell .$$

Then the above Theorem predicts:

$$d_f(A) \le \left(\frac{\ell}{c}\right)^{\frac{m}{2}} \lambda^{\frac{m}{2}} = \sqrt{6}\,\lambda^{1/2}$$