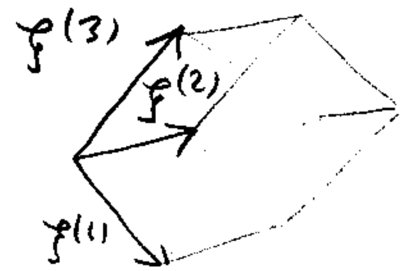
(8.2) Evolution of n-dimensional volumes

Idea: Find the smallest n such that each n-dimensional volume is decreasing.
Then $d_f(\mathcal{A}) \le n$.

Volumes: Let $\xi^{(1)}, \dots, \xi^{(n)} \subseteq \mathbb{R}^k$ linear independent vectors which span a parallelepiped

$$\xi^{(1)} \wedge \dots \wedge \xi^{(n)}$$



We denote the volume by $|\xi^{(1)} \wedge \dots \wedge \xi^{(n)}|$

Let m be the matrix made out of the vectors $\xi^{(i)}$:

$$m = \begin{pmatrix} | & | & & | \\ \xi^{(1)} & \xi^{(2)} & \cdots & \xi^{(n)} \\ | & | & & | \end{pmatrix}$$

Let $M(\xi^{(1)}, \dots, \xi^{(n)})$ be the matrix with components

$$M_{ij} = \xi^{(i)} \cdot \xi^{(j)}$$

Then we have

Lemma 1

(i) $m^T m = M$

(ii) $|\xi^{(1)} \wedge \dots \wedge \xi^{(n)}| = |\det m| = \sqrt{|\det M|}$

(iii) M has real positive eigenvalues

(iv) $\log(\det M) = \operatorname{tr}(\log M)$

(v) If $\xi^{(1)}(t), \dots \xi^{(n)}(t)$ depend on time, then

$$\frac{d}{dt} \operatorname{tr}(\log M(t)) = \operatorname{tr}\left(M(t)^{-1} \frac{dM(t)}{dt}\right)$$

Proof:

$$m = \begin{pmatrix} | & | & & | \\ \xi^{(1)} & \xi^{(2)} & \dots & \xi^{(n)} \\ | & | & & | \end{pmatrix}$$

$$m^T = \begin{pmatrix} — \xi^{(1)} — \\ \\ — \xi^{(n)} — \end{pmatrix}$$

$$m^T m = \begin{pmatrix} \xi^{(1)} \xi^{(1)} & \xi^{(1)} \xi^{(2)} & \dots & \xi^{(1)} \cdot \xi^{(n)} \\ & \cdot & & \\ \xi^{(n)} \xi^{(1)} & \cdot & & \xi^{(n)} \xi^{(n)} \end{pmatrix}$$

$$(m^T m)_{ij} = \xi^{(i)} \xi^{(j)} = M_{ij} \quad \Rightarrow (i)$$

$\Rightarrow M$ is symmetric, hence it has real eigenvalues λ_k with eigenvectors $\varphi^{(k)}$.

Then

$$\lambda_k = \varphi^{(k)T} M \varphi^{(k)} = \varphi^{(k)T} m^T m \varphi^{(k)}$$

$$= (m\varphi^{(k)})^T (m\varphi^{(k)}) = |m\varphi^{(k)}|^2 \geq 0 \quad \Rightarrow (iii)$$

m is the coordinate transformation that maps the canonical basis $\{e^{(1)}, \dots, e^{(n)}\}$ onto $\{\xi^{(1)} \dots \xi^{(n)}\}$:

$m e^{(i)} = \xi^{(i)}$. Then, using the transformation of an n-dimensional integral we find the volume

$$|\xi^{(1)} \wedge \dots \wedge \xi^{(n)}| = \int_{\xi^{(1)} \wedge \dots \wedge \xi^{(n)}} d\xi^{(1)} d\xi^{(2)} \dots d\xi^{(n)}$$

$$= \int_{e^{(1)} \wedge \dots \wedge e^{(n)}} |\det m| \, de^{(1)} \dots de^{(n)}$$

$$= |\det m|$$

Now $\det M = \det(m^T)\det m$

$$= (\det m)^2 = |\xi^{(1)} \wedge \dots \wedge \xi^{(n)}|^2 . \Rightarrow (ii).$$

To obtain (iv) we write $M = \sum_{i=1}^{n} \lambda_i \varphi^{(i)} \varphi^{(i)T}$
$\{\varphi^{(i)}\}$ ONB.
Then $\log M = \sum_{i=1}^{n} \log \lambda_i \, \varphi^{(i)} \varphi^{(i)T}$ (Linear Algebra)

$$\mathrm{tr}(\log M) = \sum_{i=1}^{n} \log \lambda_i \qquad (*)$$

Since $\det M = \prod_{i=1}^{n} \lambda_i$ we get $\log(\det M) = \sum_{i=1}^{n} \log \lambda_i$

$\Rightarrow$ (iv)

From (*) we get $\frac{d}{dt} \mathrm{tr}(\log M) = \sum_{i=1}^{n} \frac{\dot\lambda_i}{\lambda_i}$

We consider the right hand side of (v)

$$\mathrm{tr}\left(M^{-1} \frac{dM}{dt}\right) = \sum_{i=1}^{n} \varphi^{(i)T} M^{-1} \frac{dM}{dt} \varphi^{(i)}$$

$$= \sum_{i=1}^{n} \varphi^{(i)T} \Big(\sum_j \lambda_j^{-1} \varphi^{(j)} \varphi^{(j)T} \sum_k \big(\dot\lambda_k \varphi^{(k)} \varphi^{(k)T}$$

$$+ \lambda_k \dot\varphi^{(k)} \varphi^{(k)T} + \lambda_k \varphi^{(k)} \dot\varphi^{(k)T} \big) \varphi^{(i)} \Big)$$

$$= \sum_{i=1}^{n} \lambda_i^{-1} \varphi^{(i)T} \Big(\dot\lambda_i \varphi^{(i)} + \lambda_i \dot\varphi^{(i)} + \sum_k \lambda_k \varphi^{(k)} \dot\varphi^{(k)T} \varphi^{(i)} \Big)$$

$$= \sum_{i=1}^{n} \lambda_i^{-1} \dot\lambda_i + \varphi^{(i)T} \dot\varphi^{(i)} + \dot\varphi^{(i)T} \varphi^{(i)} \Big)$$

now $\varphi^{(i)T} \varphi^{(i)} = 1 \Rightarrow \frac{d}{dt} \left(\varphi^{(i)T} \varphi^{(i)} \right) = 0$

$$= \sum_{i=1}^{n} \lambda_i^{-1} \dot\lambda_i \quad \Longrightarrow (v)$$

$\square$

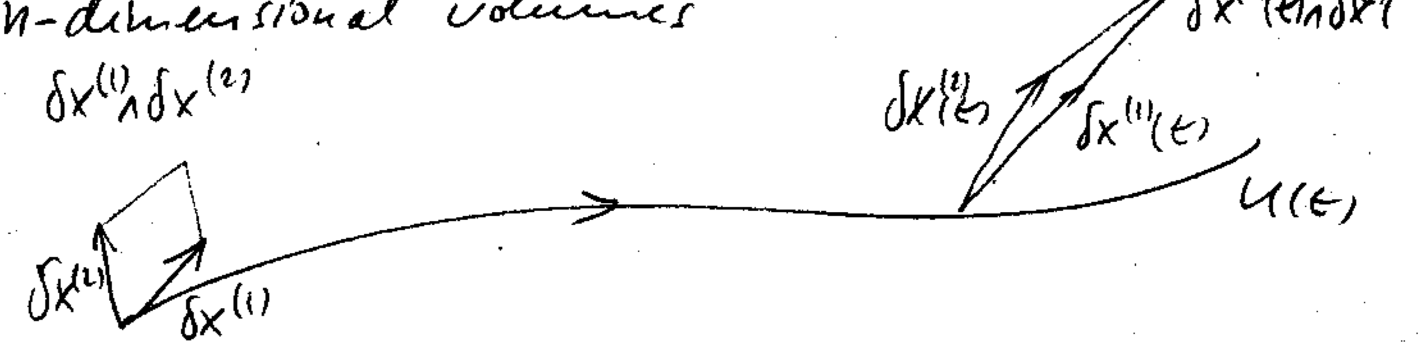
We now study the abstract problem

$$\frac{du}{dt} = F(u(t)) \quad , \quad u(0) = u_0$$

in a Hilbert-space H.

We assume there is a semigroup $S(t) u_0 = u(t)$ and a compact global attractor $\mathcal{A}$.

We are interested in the time evolution of n-dimensional volumes



We like to linearize around a given orbit $u(t)$:

<u>Definition</u>: $S(t)$ is <u>uniformly differentiable on $\mathcal{A}$</u>, if for every $u \in \mathcal{A}$ there exists a linear operator $\Lambda(t,u)$ such that for $t \geq 0$

$$\sup_{\substack{u,v \in \mathcal{A} \\ 0<|u-v|<\varepsilon}} \frac{|S(t)u - S(t)v - \Lambda(t,u)(u-v)|}{|u-v|} \xrightarrow[\varepsilon \to 0]{} 0$$

and $\sup_{u \in \mathcal{A}} \|\Lambda(t,u)\|_{op} < \infty$ for each $t \geq 0$.

$\Lambda(t,u)$ is called the <u>linearization of $S(t)$ along the orbit of u</u>

We suppose now that the linearization $\Lambda(t,u)$ is the solution semigroup of the linearized equation

$$\frac{d\mathcal{U}}{dt} = F'(S(t)u_0)\,\mathcal{U}(t), \quad \mathcal{U}(0) = \xi$$

which we write as

$$\frac{d\mathcal{U}}{dt} = L(t,u_0)\,\mathcal{U}(t) \quad \mathcal{U}(0) = \xi.$$

Let $\delta x^{(1)}_{(t)} \wedge \dots \wedge \delta x^{(n)}_{(t)}$ be an n-dimensional test volume $V_n(0) = |\delta x^{(1)}(0) \wedge \dots \wedge \delta x^{(n)}(0)|$

Let $P^{(n)}_{(t)}$ denote the orthogonal projection onto the space spanned by $\{\delta x^{(1)}(t), \dots, \delta x^{(n)}(t)\}$

(*1) <u>Note from p. 36</u>

<u>Theorem 1</u>: For t small enough we find:

$$\boxed{V_n(t) = V_n(0) \exp\left(\int_0^t \mathrm{tr}\left(L(s;u_0)\, P^{(n)}(s)\right) ds\right)}$$

<u>Proof</u>: We assume that initially $\delta x_0^{(1)} \dots \delta x_0^{(n)}$ are small. Since the solution semigroup $S(t)\delta x_0^{(i)}$ is continuous in time, $S(t)\delta x_0^{(i)}$ remain small for a certain time interval $[0, t)$.

On that interval the time evolution is given by the linearization at $u(t)$.

$$\frac{d}{dt}\delta x^{(i)}(t) = L(t;u_0)\delta x^{(i)}(t).$$

We are interested in $V_n(t) = |\delta x^{(1)}(t) \wedge \dots \wedge \delta x^{(n)}(t)|$

For that we study

$$\frac{d}{dt} \ln V_n(t) = \frac{1}{2}\frac{d}{dt} \ln V_n^2(t)$$

$$= \frac{1}{2}\frac{d}{dt} \ln\left(\det M(t)\right).$$

Where $\mathcal{M}(t) = \mathcal{M}(\delta x^{(1)}(t), \dots, \delta x^{(n)}(t))$.

From Lemma 1 (iv) and (v) we find

$$\frac{d}{dt} \ln\left(\det \mathcal{M}(t)\right) = \frac{d}{dt} \operatorname{tr}\left(\ln \mathcal{M}(t)\right)$$

$$= \operatorname{tr}\left(\mathcal{M}^{-1}(t) \frac{d\mathcal{M}(t)}{dt}\right)$$

Hence we get

$$\frac{d}{dt} \ln V_n(t) = \frac{1}{2} \operatorname{tr}\left(\mathcal{M}^{-1}(t) \frac{d\mathcal{M}(t)}{dt}\right). \qquad (*)$$

p.35 ← __Note__: If $\delta x^{(1)}, \dots, \delta x^{(n)} \in H$ linear independent then we can still define a $n \times n$-matrix

$$\mathcal{M}(\delta x^{(1)}, \dots, \delta x^{(n)}) \text{ by}$$

$$\mathcal{M}_{ij} = \left(\delta x^{(i)}, \delta x^{(j)}\right)_H$$

This matrix $\mathcal{M}$ has still the properties as summarized in Lemma 1. The only difference is, that m can no longer be explicitely given. But still $m : \{e^{(k)}\} \to \{\delta x^{(k)}\}$ is a transformation of a canonical set $\{e^{(k)}\} \subset H$ into $\{\delta x^{(k)}\} \subset H$.

We need to find an expression for $\frac{d\mathcal{M}}{dt}$. For that we assume that $\{\phi^{(j)}(t)\}$ is an orthonormal set in H which spans the same ~~volume as~~ subspace as span$\{\delta x^{(i)}(t)\}$. If we denote the coordinate transformation by $m(t): \{\phi^{(j)}(t)\} \to \{\delta x^{(i)}(t)\}$, then

$$m_{ij} = \left(\phi^{(i)}(t), \delta x^{(j)}(t)\right).$$

Again: $\mathcal{M} = m^T m$, $\mathcal{M}^{-1} = m^{-1}(m^T)^{-1}$

We express $L(t; u_0)$ in this new basis:

$$a_{ij} = \left(\phi^{(i)}, L\phi^{(j)}\right)$$

and also $\frac{d\mathcal{M}}{dt}$:

$$\frac{d\mathcal{M}_{ij}}{dt} = \left(\frac{d}{dt}\delta x^{(i)}, \delta x^{(j)}\right) + \left(\delta x^{(i)}, \frac{d}{dt}\delta x^{(j)}\right)$$

$$= \left(L\delta x^{(i)}, \delta x^{(j)}\right) + \left(\delta x^{(i)}, L\delta x^{(j)}\right)$$

For a general vector $u \in \text{span}\{\phi^{(j)}\}$ we have

$$u = \sum_{j=1}^{n} \phi^{(j)}\left(\phi^{(j)}, u\right)$$

$$\delta x^{(i)} = \sum_{k=1}^{n} \phi^{(k)}\left(\phi^{(k)}, \delta x^{(i)}\right)$$

$$L\delta x^{(i)} = \sum_{k=1}^{n} L\phi^{(k)}\left(\phi^{(k)}, \delta x^{(i)}\right)$$

Then

$$(L\delta x^{(i)}, \delta x^{(j)}) = \left(\sum_{k=1}^{n} L\phi^{(k)} (\phi^{(k)}, \delta x^{(i)}) \, , \sum_{l=1}^{n} \phi^{(l)} (\phi^{(l)}, \delta x^{(j)}) \right)$$

$$= \sum_{k,l} (L\phi^{(k)}, \phi^{(l)})(\phi^{(k)}, \delta x^{(i)})(\phi^{(l)}, \delta x^{(j)})$$

Similarly

$$(\delta x^{(i)}, L\delta x^{(j)}) = \sum_{k,l} (\phi^{(k)}, L\phi^{(l)})(\phi^{(k)}, \delta x^{(i)})(\phi^{(l)}, \delta x^{(j)})$$

Together we obtain

$$\frac{dM_{ij}}{dt} = \sum_{k,l} (\phi^{(k)}, \delta x^{(i)}) \left[(L\phi^{(k)}, \phi^{(l)}) + (\phi^{(k)}, L\phi^{(l)}) \right] (\phi^{(l)}, \delta x^{(j)})$$

$$= \sum_{k,l=1}^{n} m_{ki} (a_{lk} + a_{kl}) m_{lj}$$

hence

$$\boxed{\frac{dM}{dt} = m^T (a^T + a) m}$$

We use this in (*) to find

$$2 \frac{d}{dt} \ln V_n(t) = \operatorname{tr} \left(m^{-1} (m^T)^{-1} m^T (a^T + a) m \right)$$

$$= \operatorname{tr} \left(m^{-1} (a^T + a) m \right)$$

$$= \operatorname{tr} (a^T + a)$$

$$= 2 \operatorname{tr}(a) .$$

Now, let $P^{(n)}(t)$ denote the orthogonal projection onto span $\{\phi^{(i)}\}$.

$$P^{(n)}(t) = \sum_{i=1}^{n} \phi^{(i)} (\phi^{(i)}, \cdot \,)$$

Then

$$\left(L\rho^{(n)}\right)_{jj} = \left(\phi^{(j)}, L\sum_i \phi^{(i)}\left(\phi^{(i)}, \phi^{(j)}\right)\right)$$

$$= \sum_{i=1}^{n}\left(\phi^{(j)}, L\phi^{(i)}\right)\left(\phi^{(i)}, \phi^{(j)}\right) = \sum_i a_{ij}\,\delta_{ij} = a_{jj}$$

$$\cancel{tr\left(L\rho^{(n)}\right) = \sum_{i,j=1}^{n}\left(\phi^{(j)}, L\phi^{(i)}\right)\left(\phi^{(i)}, \phi^{(j)}\right) \Rightarrow \Sigma}$$

~~Since a is expressed in terms of $\phi^{(i)}$:~~

$$tr(a) = \cancel{\sum_{i=1}^{n}\left(\phi^{(i)}, L\phi^{(i)}\right)}$$

$$= \cancel{\sum_{i,j=1}^{n}\left(\phi^{(j)}, L\phi^{(i)}\right)\left(\phi^{(i)}, \phi^{(j)}\right)} = tr\left(L\rho^{(n)}\right)$$

Finally we integrate (*) from 0 to t to obtain:

$$V_n(t) = V_n(0)\exp\left(\int_0^t tr\left(L(s;u_0)\,\rho^{(n)}(s)\right)ds\right)$$

$\square$

We assume that the linear approximation, as used in the above theorem, gives a good approximation to the time evolution of the volume $V_n(t)$ for all $t \to \infty$.

We can identify an <u>asymptotic growth rate</u>

$$\limsup_{t\to\infty} \frac{1}{t}\left(\int_0^t tr\left(L(s;u_0)\rho^{(n)}(s)\right)ds\right)$$

and the <u>maximal asymptotic growth rate of n-dimensional volumes</u>

$$\mathcal{T}R_n(\mathcal{A}) = \sup_{u_0 \in \mathcal{A}} \; \sup_{P^{(n)}(0)} \; \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{tr}\left(L(s; u_0) P^{(n)}(s)\right) ds \}$$

We abbreviate with the notation for time averages:

$$\langle f(t) \rangle = \limsup_{t \to \infty} \frac{1}{t} \int_0^t f(s) ds$$

then

$$\mathcal{T}R_n(\mathcal{A}) = \sup_{u_0 \in \mathcal{A}} \; \sup_{P^{(n)}(0)} \left\langle \mathrm{tr}\left(L(s; u_0) P^{(n)}(s)\right) \right\rangle$$

If $\mathcal{T}R_n(\mathcal{A}) < 0$ then each n-dimensional infinitessimal volume will decay exponentially.

(B. Hunt 1999)

<u>Theorem</u> 2: Suppose $S(t)$ is uniformly differentiable on $\mathcal{A}$ and that there exists a t_0 such that $\Lambda(t; u_0)$ is compact for all $t \geq t_0$.

If $\mathcal{T}R_n(\mathcal{A}) < 0$, then $d_f(\mathcal{A}) \leq n$

<u>Proof</u> [R. Appendix B.]

Lemma 3 Let P_n be an n-dimensional projection to a subset of $L^2(Q)$, Q periodic domain in $\mathbb{R}^m$. Then

$$\operatorname{tr}(-\Delta P_n) \geq c\, n^{\frac{m+2}{m}}$$

Proof: Write $A = -\Delta$, eigenfunctions w_j, eigenvalues $\lambda_j \leq \lambda_{j+1}$. Let P_n be a projection to $\operatorname{span}\{\phi_1, \dots, \phi_n\}$, ϕ_j orthonormal.

$$\operatorname{tr}(A P_n) = \sum_{j=1}^{n} (\phi_j, A\phi_j)$$

$$\geq \inf_{\substack{\psi_1, \dots, \psi_n \in H \\ \text{linear indep.} \\ |\psi_j| = 1}} \sum_{j=1}^{n} (\psi_j, A\psi_j)$$

$$= \sum_{j=1}^{n} (w_j, A w_j)$$

$$= \sum_{j=1}^{n} \lambda_j (w_j, w_j) = \sum_{j=1}^{n} \lambda_j.$$

The eigenvalues of A on $L^2(Q)$ satisfy:

$$c\, j^{\frac{2}{m}} \leq \lambda_j \leq C\, j^{\frac{2}{m}}$$

for two constants $c, C > 0$

Hence

$$\operatorname{tr}(AP_n) \geq \sum_{j=1}^{n} \lambda_j \geq c \sum_{j=1}^{n} j^{2/m} \geq c \, \frac{1}{1+\frac{2}{m}} \, n^{\frac{2}{m}+1} \qquad \square$$

by induction.

$$\geq \tilde{c} \, n^{\frac{m+2}{m}}$$

<u>Remark</u>: The same result is valid for any C^2-domain $\Omega \subseteq \mathbb{R}^m$.