§7 Finite Dimensional Attractors

(7.1) Measures of dimension

A map $d: \mathcal{P}(H) \to \mathbb{R}^+$ is called "dimension" if

(i) $d(X) \le d(Y)$ if $X \subset Y$

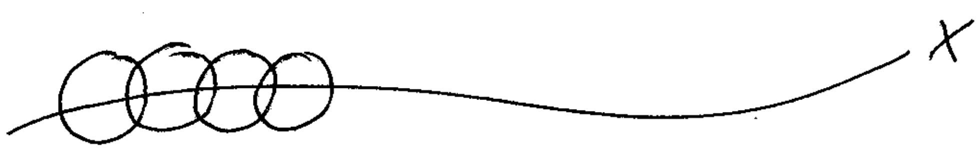
(ii) $d(X) = n$, if X is a nonempty open subset of $\mathbb{R}^n$

(iii) $d(X \cup Y) = \max(d(X), d(Y))$

The fractal dimension, $d_f(X)$:

We cover $X \subset H$ by ε-balls and count the number of balls we need as $\varepsilon \to 0$.

e.g. let $X \subset \mathbb{R}^3$ be a line



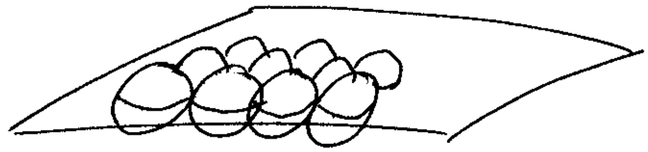
$N(X, \varepsilon) = \#$ of closed ε-balls to cover X

$N(X, \frac{\varepsilon}{2}) \cong 2 \cdot N(X, \varepsilon)$

$N(X, \frac{\varepsilon}{\gamma}) \sim \gamma \cdot N(X, \varepsilon)$

hence $N(X, \varepsilon) \sim \frac{1}{\varepsilon}$

$X \subset \mathbb{R}^3$ sphere



$N(X, \varepsilon) \sim \frac{1}{\varepsilon^2}$

$X \subset \mathbb{R}^3$ volume : $N(X, \varepsilon) \sim \frac{1}{\varepsilon^3}$

The exponent of $N(X, \varepsilon)$ as $\varepsilon \to 0$ seems to be a good notion for dimension:

<u>Definition</u> Let $X \subset H$, $\bar{X}$ compact. Then the <u>fractal dimension</u> of X, $d_f(X)$, is defined by

$$d_f(X) = \limsup_{\varepsilon \to 0} \frac{\ln N(X, \varepsilon)}{-\ln \varepsilon},$$

where $d_f(X) = +\infty$ is allowed.

Assume $N(X, \varepsilon) \approx \varepsilon^{-\gamma}$

$$\ln N(X, \varepsilon) \sim -\gamma \ln \varepsilon$$

$$\frac{\ln N(X, \varepsilon)}{-\ln \varepsilon} \sim \gamma$$

If $d > d_f(X)$, then by definition

$$N(X, \varepsilon) \le \varepsilon^{-d} \quad \text{for } \varepsilon \to 0. \qquad (*)$$

<u>Lemma 1</u> d_f is a dimension.

<u>Proof</u>: (i) If $X \subset Y$ then $N(X, \varepsilon) \le N(Y, \varepsilon)$

$\Rightarrow \; d_f(X) \le d_f(Y)$

(ii) $X \subset \mathbb{R}^n$ nonempty and open, then

$N(X, \varepsilon) \sim \frac{1}{\varepsilon^n}$

(iii) ~~$N(X \cup Y, \varepsilon) = \max$~~

If $d(X) = d(Y)$, then $d(X \cup Y) = d(X) = d(Y)$.

If $d(X) > d(Y)$, then $d(X \cup Y) = d(X)$.

(see Lemma 2)

<u>Lemma 2</u> (Properties of fractal dimension)

(i) finite unions:

$$d_f\left(\bigcup_{k=1}^{m} X_k\right) \le \max_k d_f(X_k)$$

(ii) If $f: H \to H$ is Hölder continuous with exponent θ:

$$|f(x) - f(y)| \le L\,|x - y|^{\theta},$$

then

$$d_f(f(X)) \le \frac{d_f(X)}{\theta}$$

(iii) $d_f(X \times Y) \leq d_f(X) + d_f(Y)$

(iv) $d_f(\bar{X}) = d_f(X)$

<u>Proof</u> (i) We have $N\left(\bigcup_{k=1}^{n} X_k, \varepsilon\right) \leq \sum_{k=1}^{n} N(X_k, \varepsilon)$.

For all $d > \max_k d_f(X_k)$ and ε small enough we have from $(*)$ that

$$N(\cup X_k, \varepsilon) \leq n \cdot \varepsilon^{-d},$$

hence $d_f(\cup X_k) \leq d$, and consequently

$$d_f(\cup X_k) \leq \max_k d_f(X_k).$$

(ii) Let X be covered by $N(X, \varepsilon)$ balls of radius ε. The images of f have diameter at most $L\varepsilon^{\theta}$ and they cover $f(X)$. Hence

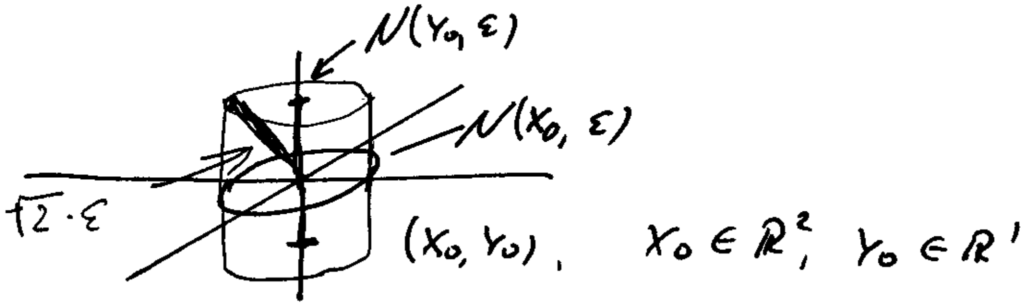
$$N(f(X), L\varepsilon^{\theta}) \leq N(X, \varepsilon).$$

Then
$$d_f(f(X)) \leq \limsup_{\varepsilon \to 0} \frac{\ln N(f(X), L\varepsilon^{\theta})}{-\ln(L\varepsilon^{\theta})}$$

$$\leq \limsup_{\varepsilon \to 0} \frac{\ln N(X, \varepsilon)}{-\ln L - \theta \ln \varepsilon}$$

$$= \frac{d_f(X)}{\theta}$$

(iii) If X is covered by $N(X,\varepsilon)$ balls of radius ε and Y is covered by $N(Y,\varepsilon)$ balls of radius ε then $X \times Y$ is covered by the product of all possible pairs, $N(X,\varepsilon)\cdot N(Y,\varepsilon)$. The radius



of those pairs is no more than 2ε:

$$N(X\times Y, 2\varepsilon) \le N(X,\varepsilon)\,N(Y,\varepsilon).$$

Then

$$\begin{aligned} d_f(X\times Y) &= \limsup_{\varepsilon\to 0} \frac{\ln N(X\times Y, 2\varepsilon)}{-\ln 2\varepsilon} \\ &\le \limsup_{\varepsilon\to 0} \frac{\ln N(X,\varepsilon)N(Y,\varepsilon)}{-\ln 2\varepsilon} \\ &= \limsup_{\varepsilon\to 0} \frac{\ln N(X,\varepsilon)}{-\ln 2-\ln\varepsilon} + \limsup_{\varepsilon\to 0} \frac{\ln N(Y,\varepsilon)}{-\ln 2-\ln\varepsilon} \\ &= d_f(X) + d_f(Y). \end{aligned}$$

(iv) If $N(X,\varepsilon)$ is the number of closed balls which cover X then these balls also cover $\bar{X}$.

Examples in class presentation #9:

1) $I_Q = [0,1] \cap \mathbb{Q}$, $d_f(I_Q) = 1$

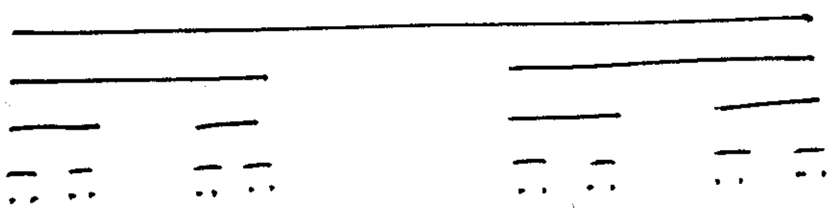
2) $G_k = \{n^{-k}\}_{n=1}^{\infty}$, $d_f(X) = \frac{1}{k+1}$

3) $\{e_n\}_{n=1}^{\infty}$ orthonormal basis in a H-space H

$H_k = \{n^{-k} e_n\}_{n=1}^{\infty}$, $d_f(H_k) = \frac{1}{k}$

$H_{log} = \{\frac{e_n}{\log n}\}_{n=1}^{\infty}$, $d_f(H_{log}) = \infty$

4) Cantor's middle third set:

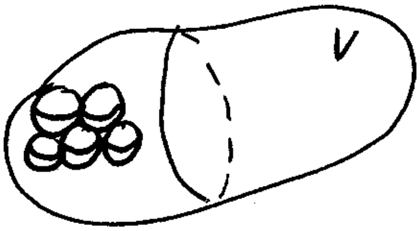


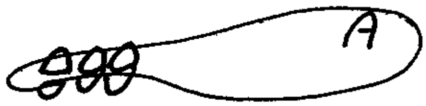
$$d_f(C) = \frac{\ln 2}{\ln 3} \cong 0.63$$

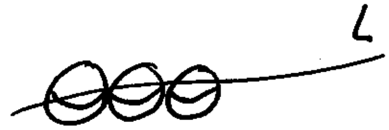
The Hausdorff dimension

We cover a d-dimensional set with Balls of radius $\leq \varepsilon$.

Volume $(d=3)$ | area $(d=2)$ | length $(d=1)$







$\mathrm{Vol}(B(x_i, r_i)) \sim r_i^{3}$ $\qquad$ $\mathrm{Vol}(B(x_i, r_i) \cap A) \sim r_i^2$ $\qquad$ $\mathrm{Vol}(B(x_i, r_i) \cap L) \sim r_i$

$\mathrm{Vol}(V) \sim \sum_i r_i^{3}$ as $r_i \to 0$ $\qquad$ $\mathrm{Vol}(A) \sim \sum_i r_i^2$ as $r_i \to 0$ $\qquad$ $\mathrm{Vol}(L) \sim \sum_i r_i$ as $r_i \to 0$

We are interested to find the index of r_i^d.

<u>Definition:</u>

1) $$\mu(X, d, \varepsilon) := \inf \left\{ \sum_i r_i^d : r_i \leq \varepsilon,\ X \subseteq \bigcup_i B(x_i, r_i) \right\}$$

2) The <u>d-dimensional Hausdorff</u> measure is defined as

$$\mathcal{H}^d(X) = \lim_{\varepsilon \to 0} \mu(X, d, \varepsilon)$$

3) The Hausdorff-dimension of X is defined as

$$d_H(X) = \inf_{d > 0} \left\{ d : \mathcal{H}^d(X) = 0 \right\}$$

Remarks

(i) $\mu(X, d, \varepsilon)$ is nonincreasing in ε.

If $\varepsilon_1 < \varepsilon_2$, then $\{ \bigcup_i B(x_i, r_i) ; r_i \le \varepsilon_1 \} \subset \{ \bigcup_i B(x_i, r_i) ; r_i \le \varepsilon_2 \}$

(ii) On $\mathbb{R}^n$: $\mathcal{H}^n \sim \lambda^n$ (Lebesgue measure)

(iii) Let M be a d-dimensional manifold. Suppose $q > d$, then

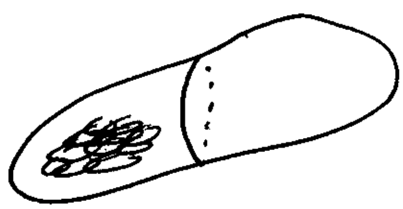
$$\sum_i r_i^q \le \underbrace{\varepsilon^{q-d}}_{\downarrow\, 0} \cdot \underbrace{\sum_i r_i^d}_{\to \mathrm{Vol}(M) \text{ as } \varepsilon \to 0} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

$$\Rightarrow \mathcal{H}^q(M) = 0.$$

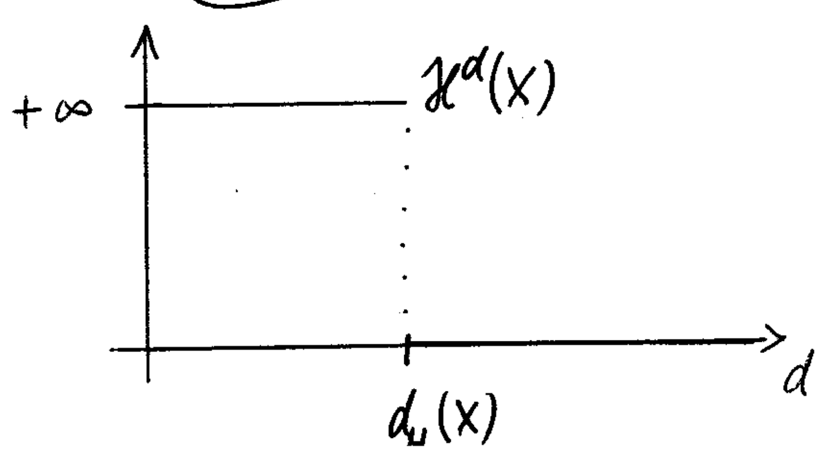
(iv) Suppose $p < d$, then

$$\sum_i r_i^p \ge \underbrace{\varepsilon^{p-d}}_{\downarrow\, +\infty} \underbrace{\sum_i r_i^d}_{< \infty} \longrightarrow +\infty \quad \text{as } \varepsilon \to 0$$

e.g. try to fill a volume V ($d=3$) with disks ($d=2$):



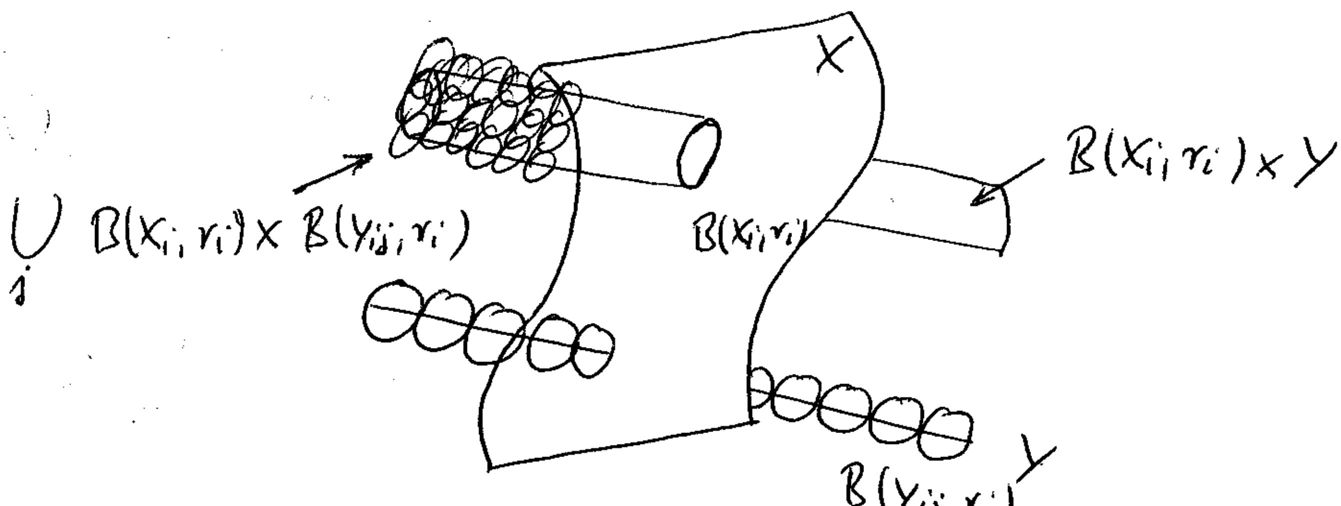
(v) given X:



If $s > d_H(X)$ we have $\mathcal{H}^s(X) = 0$, hence we can cover X with balls $\{B(x_i, r_i)\}$ such that

$$\sum_i r_i^s < 1.$$

Now, for each i we cover Y with $N(Y, r_i)$ balls of radius r_i, we call these $B(y_{ij}, r_i)$.
Then $B(x_i, r_i) \times Y$ is covered by the $N(Y, r_i)$ products of balls $B(x_i, r_i) \times B(y_{ij}, r_i)$



and so $$X \times Y \subset \bigcup_i \bigcup_j B(x_i, r_i) \times B(y_{ij}, r_i) \subset \bigcup_i \bigcup_j B((x_i, y_{ij}), 2r_i)$$

Then

$$\mu(X \times Y, s+t, 2\delta) \leq \sum_i \sum_j (2r_i)^{s+t}$$
$$\leq \sum_i N(Y, r_i)\, 2^{s+t} r_i^{s+t}$$
$$\leq 2^{s+t} \sum_i r_i^{-t} r_i^{s+t} < 2^{s+t}$$

Lemma 3: (Properties of the Hausdorff dimension)

(i) countable unions:

$$d_H\left(\bigcup_{k=1}^{\infty} X_k\right) \le \sup_k d_H(X_k)$$

(ii) If $f: H \to H$ is Hölder continuous with exponent θ, then $d_H(f(X)) \le \dfrac{d_H(X)}{\theta}$

(iii) $d_H(X \times Y) \le d_f(X) + d_H(Y)$

(iv) $d_H(X) \le d_f(X)$ fractal dimension!

Proof:

(i) Suppose that $d > \sup_k d_H(X_k)$, then $\mathcal{H}^d(X_k) = 0$.

If $\bigcup_i B_i^k$ is a covering of X_k, then $\bigcup_k \left(\bigcup_i B_i^k\right)$ covers $\bigcup_k X_k$. Hence

$$\mu\left(\bigcup_{k=1}^{\infty} X_k, d, \varepsilon\right) \le \sum_{k=1}^{\infty} \mu(X_k, d, \varepsilon).$$

and $\mu(X_k, d, \varepsilon) \le \lim_{\varepsilon \to 0} \mu(X_k, d, \varepsilon) = \mathcal{H}^d(X_k)$

Hence for each $\varepsilon > 0$ we have

$$\mu\left(\bigcup_{k=1}^{\infty} X_k, d, \varepsilon\right) \le \sum_{k=1}^{\infty} \mathcal{H}^d(X_k)$$

And also in the limit as $\varepsilon \to 0$:

$$\mathcal{H}^d\left(\bigcup_{k=1}^{\infty} X_k\right) \leq \sum_{k=1}^{\infty} \mathcal{H}^d(X_k)$$

Now, if $d > \sup_k d_H(X_k)$, then

$$\mathcal{H}^d\left(\bigcup_{k=1}^{\infty} X_k\right) \leq 0, \quad \text{hence } d_H\left(\bigcup_{k=1}^{\infty} X_k\right) \leq \sup_k d_H(X_k).$$

ii) If $B(x_i, r_i)$, $r_i \leq \varepsilon$, is a cover of X, then $B(f(x_i), L r_i^\theta)$ is a cover of $f(X)$ and $L r_i^\theta \leq L\varepsilon^\theta$. Therefore

$$\sum_i \left[L r_i^\theta\right]^{s/\theta} = L^{s/\theta} \sum_i r_i^s$$

$$\Rightarrow \mu\left(f(X), \tfrac{s}{\theta}, L\varepsilon^\theta\right) \leq L^{s/\theta} \mu(X, s, \varepsilon)$$

And as $\varepsilon \to 0$:

$$\mathcal{H}^{s/\theta}[f(X)] \leq L^{s/\theta} \mathcal{H}^s(X).$$

Finally, if $d_H(X) = d$, then $d_H(f(X)) = \frac{d}{\theta}$.

(iii) We choose $s > d_H(X)$, $t > d_f(Y)$ and we show that $d_H(X \times Y) \leq s + t$.

If $t > d_f(Y)$, then there exists a $\delta_0 > 0$ such that $N(Y, \delta) \leq \delta^{-t}$ for $\delta \geq \delta_0$.

Then $\mathcal{H}^{s+t}(X \times Y) < \infty$, hence

$$d_H(X \times Y) \leq s+t \quad \text{for all} \quad s > d_H(X), \quad t > d_f(Y).$$

(iv) We apply (iii) with $X = \{0\}$. Then

$Y \times \{0\} \cong Y$ and $d_H(\{0\}) = 0$.

$d_H(Y) \leq d_f(Y)$.

$\square$

Examples

1) $I_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$, $G_k = \{n^{-k}\}_{n=1}^{\infty}$, $H_k = \{n^{-k} e_n\}_{n=1}^{\infty}$.

$d_H(I_{\mathbb{Q}}) = d_H(G_k) = d_H(H_k) = 0$.

If X is countable, then $d_H(X) = 0$:

Proof. $X = \{X_k\}_{k=1}^{\infty}$, then from property (i) we get

$$d_H\left(\bigcup_k \{X_k\}\right) \leq \sup_k d_H(\{X_k\}) = 0.$$

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2) Cantor-set. Claim $d_H(C) \geq \frac{\ln 2}{\ln 3}$.

We show that $\mathcal{H}^s(C) \geq \frac{1}{2}$ for $s = \frac{\ln 2}{\ln 3}$.

Let C be covered by balls $B(x_i, r_i) = (x_i - r_i, x_i + r_i)$.

For each i we choose k_i such that

$$3^{-(k_i+1)} < r_i < 3^{-k_i}$$

then $B(x_i, r_i)$ intersects at most one of the intervals C_k. And for each $j > k_i$ each ball $B(x_i, r_i)$ intersects at most 2^{j-k_i} intervals C_j.

$$2^{j-k_i} = 2^j 2^{-k_i} = 2^j 3^{-sk_i} \le 2^j 3^s r_i^s$$

$$s = \frac{\ln 2}{\ln 3} \Rightarrow s \ln 3 = \ln 2$$

$$\Rightarrow 3^s = 2$$

$$3^{-sk_i} = \left(3^{-(k_i+1)} \cdot 3\right)^s \le r_i^s 3^s$$

Choosing j large enough: $3^{-(j+1)} \le r_i$ for all i.

Finally, $\bigcup_i B(x_i, r_i)$ intersects all of the 2^j intervals in C_j. We count intervals to obtain

$$2^j \le \sum_i 2^j 3^s r_i^s$$

$$\Rightarrow \sum_i r_i^s \ge 3^{-s} = \frac{1}{2},$$

hence $\mathcal{H}^s(C) \ge \frac{1}{2}$ and $d_H(C) \ge s = \frac{\ln 2}{\ln 3}$.

Fractal dimension versus Hausdorff dimension

Definition

(i) The "approximative fractal measure"

$$\mu_f(X, d, \varepsilon) = N(X, \varepsilon)\,\varepsilon^d$$

(ii) The "d-dimensional fractal measure"

$$m_f(X, d) = \limsup_{\varepsilon \to 0} \mu_f(X, d, \varepsilon)$$

Lemma 1 $\quad d_f(X) = \inf_{d>0} \{ d : m_f(X, d) = 0 \}$

Proof. $\quad D := \inf_{d>0} \{ d : m_f(X, d) = 0 \}$

$D \leq d_f(X)$: Let $e > d > d_f(X)$ ~~such that~~ there is an $\varepsilon_0 > 0$ such that $N(X, \varepsilon) < \varepsilon^{-d}$ for $\varepsilon \leq \varepsilon_0$.

Then $m_f(X, e) \leq \limsup_{\varepsilon \to 0} \varepsilon^e \varepsilon^{-d} = 0$

Hence $D < e$ for all $e > d > d_f(X)$

$\Rightarrow \; D \leq d_f(X)$.

$D \geq d_f(X)$. Let $c < d < d_f(X)$, then there is a sequence $\varepsilon_i \to 0$ such that $\dfrac{\ln N(X, \varepsilon_i)}{-\ln \varepsilon_i} > d$ for all i

$$N(X, \varepsilon_i) > \varepsilon_i^{-d}$$

Then $m_f(X, c) \geq \limsup_{i \to \infty} \varepsilon_i^c \, N(X, \varepsilon_i)$

$$\geq \limsup_{i \to \infty} \varepsilon_i^c \, \varepsilon_i^{-d}$$

$$> 1$$

Hence for every $c < d_f(X)$: $m_f(X, c) > 1$ which implies $D \geq d_f(X)$.

$\square$

Since $\mu(X, d, \varepsilon) \leq \mu_f(X, d, \varepsilon)$ we conclude again that $d_H(X) \leq d_f(X)$.

<u>Corollary</u> C Cantor set. $d_H(C) = d_f(C) = \frac{\ln 2}{\ln 3}$

<u>Proof</u>. We saw that $d_H(C) \geq \frac{\ln 2}{\ln 3}$ and $d_f(C) = \frac{\ln 2}{\ln 3}$ hence $d_H(C) = d_f(C)$.