(6.2) Injectivity

Theorem 1: $f \in H$ then the solution semigroup of the 2-D Navier-Stokes equations is injective on the attractor $\mathcal{A}_H$.

Proof: Apply Lemma

$w := u - v$, since $u, v \in \mathcal{A}$ they are strong solutions, hence $w \in L^\infty(0,T;V) \cap L^2(0,T;\mathcal{D}(A))$.

$$\frac{dw}{dt} + \nu A w = B(v,v) - B(u,u) =: h(t, w(t)).$$

$$|h(t,w(t))| = |B(v,v) - B(u,u)|$$

$$\leq |B(u,u-v)| + |B(u-v,v)|$$

Proposition $$\leq k\left(\|u\|_\infty \|w\| + |w|^{1/2} \|w\|^{1/2} \|v\|^{1/2} |Av|^{1/2} \right)$$

Now we use Agmon's inequality (Exercise 12.4):

$$\|u\|_\infty \leq C |Au|^{1/2} |u|^{1/2} \quad \text{for all } u \in \mathcal{D}(A).$$

Then $$|h(t,w(t))| \leq C\left(|u|^{1/2} |Au|^{1/2} \|w\| + |w|^{1/2} \|w\|^{1/2} |v|^{1/2} |Av|^{1/2} \right),$$

$$\leq C\left(|Au| + |Av| \right) \|w\| .$$

We can apply Theorem 3 in (5.2) to conclude injectivity. $\square$

Corollary 2: $\left(\mathcal{B}_{H}, \{S(t)\}_{t\in\mathbb{R}}\right)$ is a dynamical system.