§6 The Global Attractor of the Navier-Stokes Equations

$$u_t + \nu Au + B(u,u) = f \quad \text{in 2-D}$$

$$H = \{u \in \dot{\mathbb{L}}^2_p(Q) : \nabla \cdot u = 0\}, \quad |\cdot|$$

$$V = \{u \in \dot{\mathbb{H}}^1_p(Q) : \nabla \cdot u = 0\}, \quad \|\cdot\|$$

$$V^*, \quad \|\cdot\|_*$$

(6.1) Global Attractor

<u>Proposition 1</u>: There is an absorbing set in H for $f \in V^*$:

For each $u_0 \in H$ there exists a time $t_0(|u_0|) > 0$ and constants ρ_H, I_V such that

$$|u(t)| \le \rho_H \quad \text{and} \quad \int_t^{t+1} \|u(s)\|^2 ds \le I_V, \qquad t \ge t_0(|u_0|).$$

<u>Proof</u>: $u_t + \nu Au + B(u,u) = f$

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 + \underbrace{b(u,u,u)}_{=0} = \langle f, u\rangle$$

$$\le \|f\|_* \|u\|$$

$$\le \frac{1}{2\nu}\|f\|_*^2 + \frac{\nu}{2}\|u\|^2$$

$$\frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \frac{\|f\|_*^2}{\nu} \qquad (*)$$

Poincaré inequality: $\|u\|^2 \ge \lambda_1 |u|^2$

$$\frac{d}{dt}|u|^2 \le -\lambda_1 \nu |u|^2 + \frac{\|f\|_*^2}{\nu}$$

Gronwalls Lemma:

$$|u(t)|^2 \le |u_0|^2 e^{-\nu\lambda_1 t} + \frac{\|f\|_*^2}{\nu^2\lambda_1}\left(1-e^{-\nu\lambda_1 t}\right)$$

Choose $t_0(|u_0|) = \max\left(-\frac{1}{\nu\lambda_1}\ln\left(\frac{\|f\|_*^2}{\nu^2\lambda_1 |u_0|^2}\right),\ 0\right)$

Then

$$|u(t)|^2 \le 2\,\frac{\|f\|_*^2}{\nu^2\lambda_1} = \rho_H^2$$

Now we integrate $(*)$ from t to $t+1$:

$$\nu\int_t^{t+1}\|u(s)\|^2 ds \le \frac{\|f\|_*^2}{\nu} + \underbrace{|u(t)|^2}_{\le \rho_H} \underbrace{- |u(t+1)|^2}_{\le 0}$$

$$\le I_\nu, \quad I_\nu = \frac{\|f\|_*^2}{\nu} + \rho_H \quad \text{for } t \ge t_0(|u_0|),$$

$\square$

__Proposition 2__ $f \in H$, there exists an absorbing set in H!
For each $u_0 \in H$ there is $t_1(|u_0|) > 0$, constants ρ_V, I_A such that

$$\|u(t)\| \le \rho_V, \quad \int_t^{t+1}|Au(s)|^2 ds \le I_A \quad \text{for } t \ge t_1(|u_0|).$$

Proof: $$\frac{d}{dt}\|u\|^2 + \nu|Au|^2 \le \frac{|f|^2}{\nu} \qquad \text{(using Young)} \qquad (**)$$

$$\Rightarrow \frac{d}{dt}\|u\|^2 \le \frac{|f|^2}{\nu}.$$

We integrate from s to t, $s > t-1$:

$$\|u(t)\|^2 \le \|u(s)\|^2 + \frac{|f|^2}{\nu}$$

We integrate again from $s = t-1$ to $s = t$:

$$\|u(t)\|^2 \le \int_{t-1}^{t} \|u(s)\|^2 ds + \frac{|f|^2}{\nu}$$

$$\le \underbrace{I_\nu + \frac{|f|^2}{\nu}}_{\rho_\nu} \qquad \text{for } t > t_0(|u_0|) + 1 = t_1 \qquad (2.1)$$

Finally we integrate $(**)$ from t to $t+1$ to obtain

$$\nu \int_t^{t+1} |Au|^2 ds \le \frac{|f|^2}{\nu} + \|u(t)\|^2 \le \frac{|f|^2}{\nu} + \rho_\nu =: I_A. \qquad \square$$

and ~~with~~ (2.2)

Theorem 3: There exists a compact global attractor $\mathcal{A} \subset H$. Moreover, solutions on the attractor are strong.

Proof: From Proposition 2 it follows that with initial data $u_0 \in H$ we have

$$u \in L^\infty(0,T; V) \cap L^2(0,T; \mathcal{D}(A)),$$

hence strong solutions.

Proposition 4: If f is independent of time, then we find an absorbing set in $\mathcal{D}(A)$

$$|Au(t)| \le \rho_A \quad , \quad t \ge t_2(|u_0|)$$

Theorem 5: Compact global attractor in $\mathcal{A}_V \subset V$.

Lemma 6: $\mathcal{A}_V = \mathcal{A}_H$.

Theorem 6: $f \in \dot{C}_p^\infty(Q) \Longrightarrow \mathcal{A} \subseteq \dot{C}_p^\infty(Q)$.