(5.4) The Chaffee-Infante Equation

On $[0,\pi]$ we study
$$\left.\begin{aligned} u_t - u_{xx} &= \lambda(u-u^3) \\ u(0) = u(\pi) &= 0 \end{aligned}\right\} \quad (1)$$

<u>Lemma 1</u>: For $0<\lambda<1$ we have $\mathcal{A} = \{0\}$.

<u>Proof</u>:
$$\frac{1}{2}\frac{d}{dt}|u|^2 + \|u\|^2 = \lambda \int |u|^2 - |u|^4\, dx$$
$$\leq \lambda |u|^2$$

Poincaré inequality on $[0,\pi]$: $|u| \leq \|u\|$, then
$$\frac{1}{2}\frac{d}{dt}|u|^2 \leq (\lambda-1)|u|^2$$
hence $|u(t)| \longrightarrow 0$ for $t\to\infty$, if $\lambda<1$. $\square$

For larger λ the attractor becomes more complicated.

<u>Steady states</u> $\quad -u_{xx} = \lambda(u-u^3)$, $\quad u(0)=u(\pi)=0$.

$$\left.\begin{aligned} u_x &= v \\ v_x &= -\lambda(u-u^3) \end{aligned}\right\} \quad (2)$$

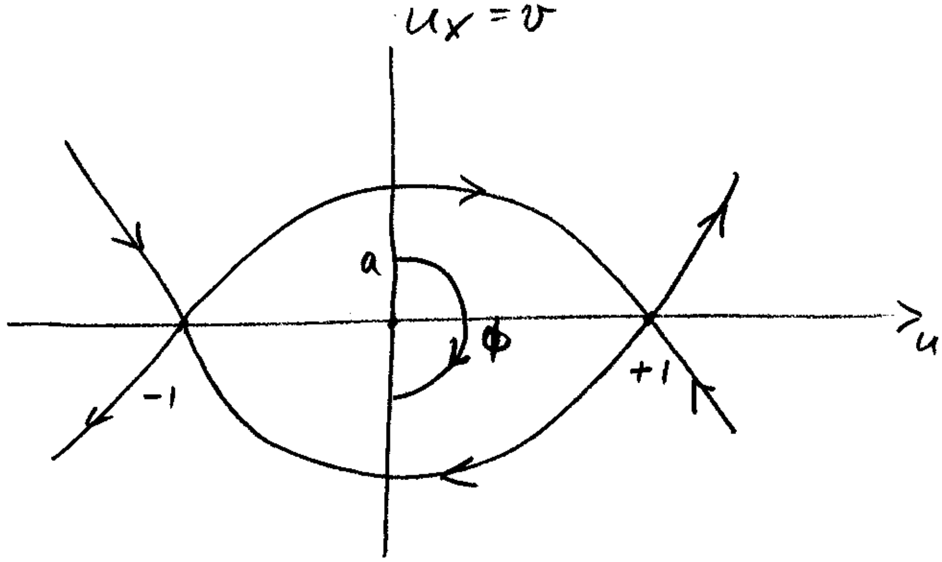
(2) has equilibria $v=0$, $u=0, \pm 1$

Linearization shows: $(-1,0)$, $(+1,0)$ are saddle points

$(0,0)$ is a center.

Hamilton function $H(u,v) = \frac{1}{2}v^2 + \lambda\left(\frac{1}{2}u^2 - \frac{1}{4}u^4\right)$

$\frac{d}{dt}H(u,v) = 0 \implies H(u,v)$ is constant along trajectories of (2). We write $H(u,v) = \frac{\lambda}{2}E$.



Boundary conditions: $u(0)=0$, $u(\pi)=0$. Solutions are inside the region spanned by the heteroclinic orbits. The heteroclinic orbits have the energy:

$$H(1,0) = \lambda\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\lambda}{4} \implies E = \frac{1}{2}$$

$$H(0,0) = 0 \implies E = 0.$$

Candidates are at energies $< \frac{1}{2}$.

A solution needs exactly "time" π to connect the u_x-axis with itselfs.

For $v \geq 0$, $u_x > 0$ hence $u(x)$ is monotone.

By symmetry the trajectory crosses the u-axis at $\left(u(\frac{\pi}{2}), v(\frac{\pi}{2})\right) = (\phi, 0)$. Since from $(0, v_0)$ to $(\phi, 0)$ $u(x)$ is monotone we can use the implicit function theorem and find

$$\frac{dx}{du} = \frac{1}{u_x} \quad \text{or}$$

$$dx = \frac{1}{u_x}\, du$$

$$l = \int_0^{\phi} dx = \int_0^{\phi} \frac{1}{u_x}\, du \tag{8}$$

$$H(u,v) = \frac{\lambda}{2} E = \frac{1}{2} v^2 + \lambda\left(\frac{1}{2}u^2 - \frac{1}{4}u^4\right)$$

$$\lambda E = u_x^2 + \lambda\left(u^2 - \frac{1}{2}u^4\right)$$

$$u_x = \sqrt{\lambda\left(E - u^2 + \frac{1}{2}u^4\right)}$$

We find E from the start-point $(0, u_x(0)) = (0, a)$

$$H(0,a) = \frac{1}{2}a^2 = \frac{\lambda}{2} E \Rightarrow \lambda E = a^2$$

$$u_x = \sqrt{a^2 - \lambda\left(u^2 - \frac{1}{2}u^4\right)}$$

Using (3) we find that the length of a monotone piece of a trajectory is given by

$$l = \int_0^{\phi} \frac{1}{\sqrt{a^2 - \lambda(u^2 - \frac{1}{2}u^4)}}\, du = \int_0^{\phi} \frac{1}{\sqrt{\lambda(E - u^2 + \frac{1}{2}u^4)}}\, du$$

As $E \to \frac{1}{2}$: $\quad l = \int_0^1 \frac{1}{\sqrt{2\lambda}(1-u^2)}\, du = \frac{1}{\sqrt{2\lambda}} \ln\left|\frac{1+u}{1-u}\right| \Big|_0^1 = \infty$

As $E \to 0$: $\quad l = \int_0^{\phi} \frac{1}{\sqrt{\lambda}u(\frac{1}{2}u^2 - 1)}\, du = \frac{\pi}{2\sqrt{\lambda}}$

$l(E)$ strictly increasing: Since $\lambda E = u_x^2 + \lambda(u^2 - \frac{1}{2}u^4)$ we have always $\lambda E > u^2 - \frac{1}{2}u^4$, hence the integrant is increasing in E.

The minimum length of a nontrivial solution of (2) is $l^* = \frac{\pi}{\sqrt{\lambda}}$

$2l(E)$ is a half circle. Solutions to Dirichlet conditions satisfy $n \cdot 2l(E) = \pi$ for some $n \in \mathbb{N}$

$$\frac{\pi}{\sqrt{\lambda}} \leq 2l(E) \leq \infty$$

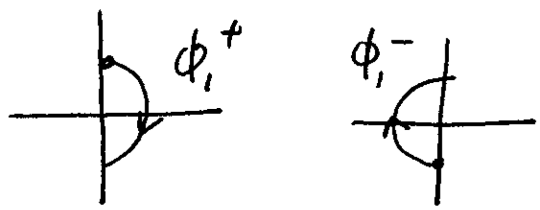
(1) If $\lambda < 1$, then $\frac{\pi}{\sqrt{\lambda}} > \lambda$ hence no $n \in \mathbb{N}$ satisfies $n2l(E) = \pi$ $\Rightarrow$ only $(0,0)$.

(ii) There are nontrivial solutions $n2l(\Theta)=\pi$, if

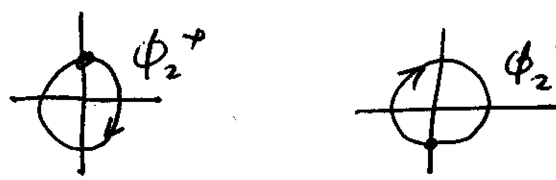
$$\pi \geq n l^* = n\frac{\pi}{\sqrt{\lambda}} \iff n \leq \sqrt{\lambda} \text{ or } \boxed{n^2 \leq \lambda}$$

Bifurcation points at $\lambda = 2^2, 3^2, 4^2, 5^2$ etc.

For $1<\lambda<2^2$ we have $1\cdot\frac{\pi}{\sqrt{\lambda}} < \pi$, there are two half loops



(iii) For $2^2<\lambda<3^2$ we have $2\cdot\frac{\pi}{\sqrt{\lambda}} < \pi$, there are two additional solutions: full loops



(iv) <u>Proposition</u>: $n^2<\lambda<(n+1)^2$ then there are $2n+1$ fixed points of (1), which is $\phi_0=(0,0)$ and n pairs $\phi_1^\pm, \dots, \phi_n^\pm$, where $\phi_j^\pm$ has j zeros in $(0,\pi)$.

Theorem (10.13) $\Rightarrow$ $\mathcal{A} = \bigcup_{z\in\mathcal{E}} \overline{W^u(z)}$,

$\mathcal{E} = \{\phi_0, \phi_1^\pm, \dots, \phi_n^\pm\}$

Linear stability analysis

$u_t = u_{xx} + \lambda(u - u^3)$ has steady state $\bar{u}$.

Set $u = \bar{u} + \mathcal{U}$, where $\mathcal{U}$ is a small perturbation and linearize:

$$\mathcal{U}_t = \mathcal{U}_{xx} + \lambda(1 - 3\bar{u}^2)\mathcal{U}$$

The eigenvalues of the operator of the right hand side determine the stability: $\left(\mathcal{U}(t,x) \to e^{\sigma t}\mathcal{U}(x)\right)$

$$\sigma \mathcal{U} = \mathcal{U}_{xx} + \lambda(1 - 3\bar{u}^2)\mathcal{U}$$

At $\bar{u} = 0$ we get $\quad \sigma\mathcal{U} = \mathcal{U}_{xx} + \lambda\mathcal{U}$

or $$\left.\begin{aligned} &\mathcal{U}_{xx} = (\sigma - \lambda)\mathcal{U} \\ &\mathcal{U}(0) = 0, \ \mathcal{U}(\pi) = 0 \end{aligned}\right\}$$

This eigenvalue problem has been solved (Assignments). e-functions $\mathcal{U}_n = \sin(nx)$

e-values $-n^2$.

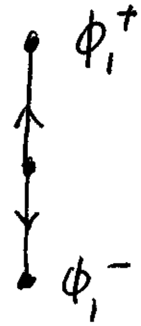
Hence $\sigma_n - \lambda = -n^2 \Rightarrow \boxed{\sigma_n = \lambda - n^2}$

$\lambda < 1$: $(0,0)$ is asymptotically stable.

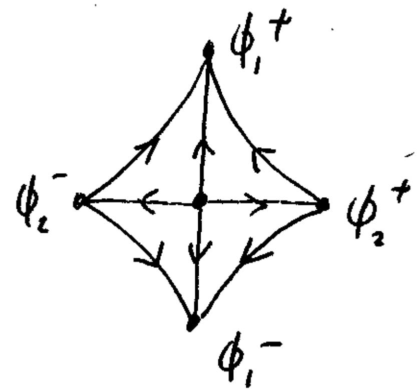
Each time as λ passes some n^2 we loose one stable eigenvalue. We obtain saddle-node bifurcations.

$\mathcal{C}_\lambda$:

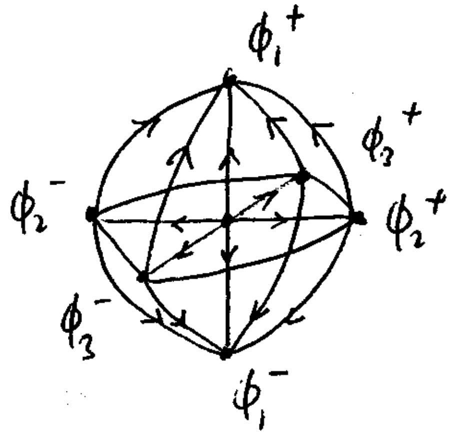
$\lambda < 1$



$1 < \lambda < 4$



$4 < \lambda < 9$



$9 < \lambda < 16$