(5.2) Injectivity

Lemma 1: $V \subset\subset H \cong H^* \subset V^*$. Assume

$w \in L^\infty(0,T;V) \cap L^2(0,T;\mathcal{D}(A))$ satisfies

$$\frac{dw}{dt} + Aw = h(t, w(t)) \quad \text{in the } L^2(0,T;H)\text{-sense}$$

Assume $A: V \to V^*$ is bounded,

$$|h(t,w(t))| \le k(t)\,\|w(t)\| \quad , \quad k(t) \in L^2(0,T).$$

Then $\Lambda(t) := \dfrac{\|w(t)\|^2}{|w(t)|^2}$ satisfies

$$\Lambda(t) \le \Lambda(0) \exp\left(\int_0^t k^2(s)\,ds \right)$$

Proof. In class presentation #7

Theorem 2 (backward uniqueness) Let $w(t)$ satisfy the assumptions of the previous Lemma. If $w(T) = 0$ for some $T > 0$, then $w(t) = 0$ for all $0 \le t \le T$.

Proof: By contradiction. Suppose $w(t_0) \neq 0$ for some $t_0 \in [0,T)$. Since $w, \frac{dw}{dt} \in L^2(0,T;L^2)$ we have (by Corollary 7.3) $w \in C^0([0,T]; H')$. By continuity there is an interval $[t_0, t_1)$ such that

$|w(t)| \neq 0$ for $t \in [t_0, t_1)$ and $w(t_1) = 0$.

On this interval we study $t \mapsto -\ln |w(t)|$

$$\frac{d}{dt} \ln \frac{1}{|w(t)|} = \frac{d}{dt}\left(-\frac{1}{2}\right) \ln |w(t)|^2$$

$$= -\frac{1}{2} \, \frac{2(w', w)}{|w|^2}$$

$$= - \frac{(h - Aw, w)}{|w|^2}$$

$$= \Lambda - \frac{(h, w)}{|w|^2} \leq \Lambda + k \Lambda^{1/2}$$

Young

$$\leq \Lambda + \frac{1}{2} k^2 + \frac{1}{2} \Lambda = \frac{3}{2} \Lambda + \frac{1}{2} k^2$$

Integrate between $[t_0, t)$: $t < t_1$

$$\ln \frac{1}{|w(t)|} = \ln \frac{1}{|w(t_0)|} + \underbrace{\int_{t_0}^{t} \left(\frac{3}{2}\Lambda + \frac{1}{2} k^2\right) ds}$$

bounded for all $t \leq T$
from the previous Lemma 1.

$$\Rightarrow \ln \frac{1}{|w(t)|} < \infty \text{ for all } t \leq T$$

$\lightning$ contradiction.

<u>Theorem 3</u> (Injectivity of R-D-eq on the attractor)
The solution semigroup $S(t)$ of the R-D-eq is injective on $\mathcal{A}$. i.e. if $u(t)$ and $v(t)$ are two solutions with $u(T) = v(T)$ for some $T > 0$ then $u(t) = v(t)$ for all $0 \le t \le T$.

<u>Proof</u>. The equation for $w = u - v$ is

$$\frac{dw}{dt} + Aw = f(u) - f(v)$$

Since the attractor $\mathcal{A}$ is bounded in V and in H^2 (Theorem 5 in (5.1)) we have

$$w \in L^\infty(0,T; V) \quad , \quad w \in L^2(0,T; \mathcal{D}(A))$$

We study $h(t, w(t)) := f(u(t)) - f(v(t))$

$$|f(u) - f(v)|^2 = \int_\Omega |f(u) - f(v)|^2 dx$$

$$= \int_\Omega \left| \int_{v(x)}^{u(x)} f'(s) ds \right| dx$$

$$\le \|f'\|_\infty^2 \; |u - v|_{L^2}^2$$

$$\le \ell^2 |u - v|^2$$

Hence $|h(t, w(t))| \le \ell \; |w(t)| \le c \|w(t)\|$

Poincaré inequality.

We apply Lemma 1 and Theorem 2 to obtain $w(t) = 0$ for all $0 \le t \le T$.

$\square$

<u>Corollary 4</u>: $(\mathcal{A}, \{S(t)\}_{t \in \mathbb{R}})$ where $\mathcal{A}$ is equipped with L^2-norm forms a <u>dynamical system</u>