§5 The global attractor for R-D equations

$$u_t - \Delta u = f(u), \quad u|_{\partial\Omega} = 0$$

$$-k - \alpha_1 |u|^p \le f(u)u \le k - \alpha_2 |u|^p$$

$$f'(s) \le l$$

$H = L^2$, $V = H_0^1$, $V^* = H^{-1}$, norms $|\cdot|$, $\|\cdot\|$, $\|\cdot\|_*$

(5.1) Absorbing Sets and the Attractor

Proposition 1 : (Absorbing set in L^2)

There is a constant ρ_H, a time $t_0(|u_0|)$ such that for the solution $S(t)u_0 = u(t)$.

$$|u(t)| \le \rho_H \quad \text{for all} \quad t \ge t_0(|u_0|).$$

Moreover

$$\int_t^{t+1} \|u(s)\|^2 \, ds \le I_V, \quad t \ge t_0(|u_0|), \quad I_V > 0 \text{ const.}$$

Proof: In class presentation # 6

<u>Proposition 2:</u> If $u(t)$ is a strong solution, then there exists an absorbing set in H_0^1. There is a constant ρ_V and a time $t_1(|u_0|)$ such that

$$\|u(t)\| \leq \rho_V \quad \text{for all} \quad t \geq t_1(|u_0|).$$

<u>Proof</u>: We multiply the R-D-eq by Au and integrate:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + |Au|^2 = (Au, f(u))$$

$$= (\nabla u, f'(u)\nabla u)$$

$$\leq \ell \|u\|^2$$

We integrate between s and t, $(t-1 \leq s \leq t)$:

$$\frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|u(s)\|^2 + \int_s^t |Au|^2 d\xi \leq \ell \int_s^t \|u(\xi)\|^2 d\xi$$

$$\|u(t)\|^2 \leq 2\ell \int_s^t \|u(\xi)\|^2 d\xi + \|u(s)\|^2$$

Now we integrate this with respect to s over $(t-1, t)$.

$$\|u(t)\|^2 \leq 2\ell \int_{t-1}^t \int_s^t \|u(\xi)\|^2 d\xi\, ds + \int_{t-1}^t \|u(s)\|^2 ds$$

$$\leq 2\ell \int_{t-1}^t \int_{t-1}^t \|u(\xi)\|^2 d\xi\, ds + \int_{t-1}^t \|u(s)\|^2 ds$$

$$= (2\lambda+1)\int_{t-1}^{t} \|u(s)\|^2 ds$$

And with use of proposition 1:

$$\|u(t)\|^2 \leq \underbrace{(2\lambda+1)\,I_V}_{\varrho_V} \quad \text{for all } t \geq \underbrace{t_0(|u_0|)+1}_{t_1(|u_0|)}.$$

$\square$

<u>Corollary 3</u>: $B_V := \{\|u\| \leq \varrho_V\}$ is also absorbing set for weak solutions

<u>Proof</u>: Do the same estimates as in Prop. 2 for the Galerkin approximation and then pass to the limit $n \to \infty$.

<u>Theorem 4</u>: The above reaction-diffusion equation has a compact global attractor $\mathcal{A}$ in $L^2(\Omega)$.

<u>Proof</u>: $B_V \subset H_0^1 \subset\subset L^2$, $\omega(B_V) = \mathcal{A}$

$\square$.

Theorem 5: (Higher Regularity)

a) $\mathcal{A}$ is uniformly bounded in $L^\infty(\Omega)$:

$$\|u\|_\infty \leq \left(\frac{k}{\alpha_2}\right)^{1/p} \quad \text{for all } u \in \mathcal{A}$$ [R. Thm 11.6 p. 292]

b) $\mathcal{A}$ is uniformly bounded in $H^2(\Omega)$

[R. Thm 11.7, p. 295]

c) If in addition Ω is C^∞ and $f \in C^\infty(\mathbb{R})$, then $\mathcal{A}$ is a bounded subset of $H^k(\Omega)$ for each $k \geq 0$. If $u \in \mathcal{A}$ then $u \in C^\infty(\bar{\Omega})$.

[R. Thm 11.8, p. 295]