(4.4) Continuous dependence on parameters

Assume $S_\eta(t)$ are semigroups with attractor $\mathcal{A}_\eta$.
Assume S_η depend continuously on η. $\eta \in [0, \eta_0)$

Q: $\mathcal{A}_\eta \to \mathcal{A}_0$ for $\eta \to 0$ is some sense?

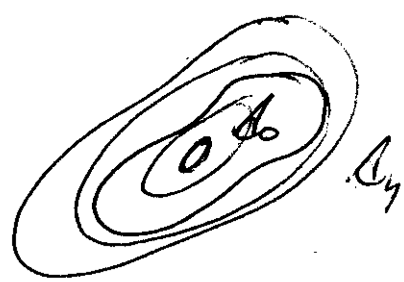
Theorem 1 (Upper Semicontinuity)
Assume there is a bounded set X : $\bigcup_{0 \le \eta \le \eta_0} \mathcal{A}_\eta \subset X$.
If for each bounded set $Y \subset H$

$$\sup_{u_0 \in Y} |S_\eta(t) u_0 - S_0(t) u_0| \to 0 \quad \text{as } \eta \to 0$$

(i.e. S_η converges uniformly on bounded sets)
then

$$\text{dist}(\mathcal{A}_\eta, \mathcal{A}_0) \to 0 \quad \text{as } \eta \to 0$$

vaguely: $\boxed{\text{" } \lim_\eta \mathcal{A}_\eta \subset \mathcal{A}_0 \text{ "}}$



Proof: Since $\mathcal{A}_0$ attracts X there is a time t^* such that $S_0(t) X \subset \mathcal{N}(\mathcal{A}_0, \frac{\varepsilon}{2})$, $t \ge t^*$.
Since S_η converges uniformly on bounded sets

$$\sup_{x \in X} |S_\eta(t) x - S_0(t) x| \le \frac{\varepsilon}{2} \quad \text{for } \begin{array}{l} 0 \le \eta < \eta^* \\ t \ge t^* \end{array}$$

Now $\mathcal{A}_\eta = S_\eta(t)\mathcal{A}_\eta \subset S_\eta(t) X \subset N(\mathcal{A}_0, \varepsilon)$

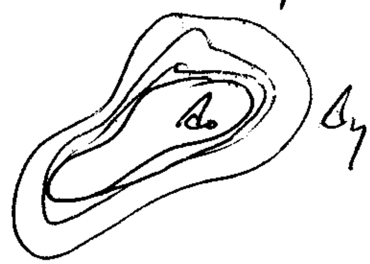
$$\Rightarrow \quad \operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}_0) \le \varepsilon \qquad \eta < \eta^*, \; t \ge t^*.$$

$\square$

<u>Lower semicontinuity</u> $\quad \operatorname{dist}(\mathcal{A}_0, \mathcal{A}_\eta) \longrightarrow 0$

" $\mathcal{A}_0 \subset$ " $\lim_\eta \mathcal{A}_\eta$ "

<u>Not</u> true in general.



<u>Theorem 2</u> (Stuart + Humphries 1996, lower semicontinuity). In addition to the assumptions of Theorem 1 we assume that

$$\mathcal{A}_0 = \bigcup_{z \in \mathcal{E}} \overline{W^u(z)}, \qquad \mathcal{E} = \text{finite number of fixed points.}$$

And we assume that the unstable manifolds vary continuously in η near $\eta = 0$, near the fixed points (S_0 to be structurally stable is sufficient). Then $\mathcal{A}_0$ is lower semicontinuous

$$\operatorname{dist}(\mathcal{A}_0, \mathcal{A}_\eta) \longrightarrow 0 \quad \text{as } \eta \to 0.$$

Together with Theorem 1 it follows that $\mathcal{A}_0$ is continuous in the Hausdorff-metric:

$$\operatorname{dist}_H(\mathcal{A}_\eta, \mathcal{A}_0) = \operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}_0) + \operatorname{dist}(\mathcal{A}_0, \mathcal{A}_\eta)$$
$$\longrightarrow 0. \quad \text{as } \eta \to 0.$$

<u>Proof</u>. $\operatorname{dist}(\mathcal{A}_0, \mathcal{A}_\eta) < \varepsilon$ for $0 \le \eta < \eta^*$:

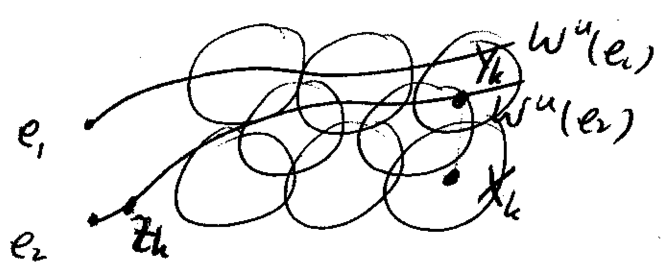
$\iff \sup_{x \in \mathcal{A}_0} \inf_{y \in \mathcal{A}_\eta} |x - y| < \varepsilon$ — " —

$\iff$ for each $x \in \mathcal{A}_0$ there is a $y \in \mathcal{A}_\eta : |x - y| < \varepsilon$.

Since $\mathcal{A}_0$ is compact, it is covered by a finite number of ε-balls $N(x_{k_i}\, \varepsilon)$ $k = 1, .., N$

Since $\mathcal{A}_0$ is the closure of the union of unstable mfds $|x_k - y_k| \le \frac{\varepsilon}{2}$, $y_k \in W^u(e_k)$.

Find z_k near e_k: $y_k = S_0(t_k)\, z_k$



Now choose $\delta > 0$ such that $|z_k - u| \le \delta$

implies $|S_0(t_k) z_k - S_0(t_k) u| \le \frac{\varepsilon}{4}$, $k = 1 .., N$

For $0 \le \eta < \eta^*$ we get $|S_0(t_k)u - S_\eta(t_k)u| < \frac{\varepsilon}{4}$ for $u \in N(A, \delta)$.

Since W^u_η depends continuously on η near the fixed points e_k, we find z_k^η near z_k, $z_k^\eta \in W^u_\eta(e$

$$|z_k^\eta - z_k| < \delta$$

Then
$$|S_\eta(t_k) z_k^\eta - y_k| = |S_\eta(t_k) z_k^\eta - S_0(t_k) z_k|$$

$$\le |S_\eta(t_k) z_k^\eta - S_0(t_k) z_k^\eta| + |S_0(t_k) z_k^\eta - S_0(t_k) z_k|$$

$$\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

$$|S_k(t_k) z_k^\eta - x| \le \varepsilon$$

$\square$