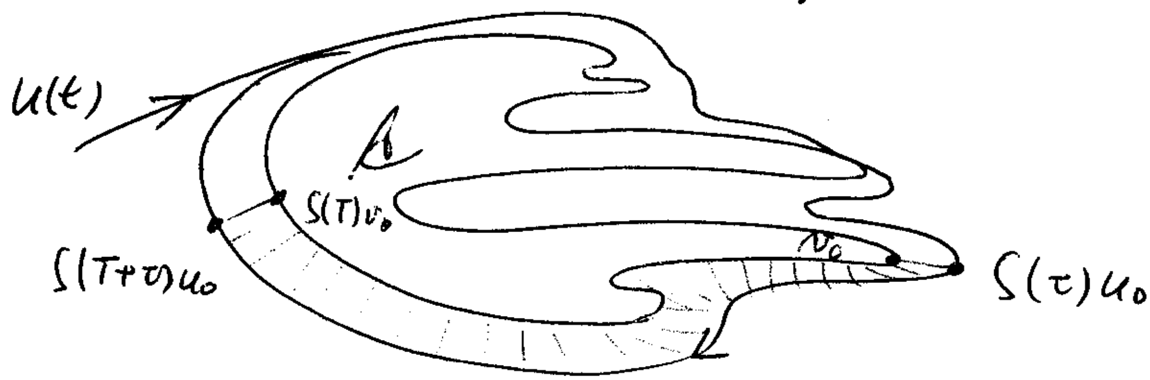
(4.3) Shadowing

Theorem 1 Given $\varepsilon > 0$, $T > 0$, $u(t) = S(t)u_0$. Then there exists $v_0 \in \mathcal{A}$ and $\tau = \tau(\varepsilon, T)$ such that

$$|u(t+\tau) - S(t)v_0| \le \varepsilon \quad \text{for all } 0 \le t \le T.$$



Proof: Since $S(t)u_0$ depends continuously on the initial conditions there exists $\delta = \delta(\varepsilon, T)$:

$$\left.\begin{array}{l} \text{If } |u_0 - v_0| \le \delta(\varepsilon, T) \text{ then } |u(t) - v(t)| \le \varepsilon \text{ for} \\ 0 \le t \le T. \end{array}\right\} \quad (*)$$

Since $\mathcal{A}$ is a attractor there exists $\tau = \tau(\delta)$:

$$\text{dist}\,(u(\tau), \mathcal{A}) \le \delta$$

Choose $v_0 \in \mathcal{A}$: $|u(\tau) - v_0| \le \delta$ then from $(*)$:

$$|u(t+\tau) - v(t)| < \varepsilon \quad \text{for all } 0 \le t \le T,$$

$$\tau = \tau(\delta) = \tau(\varepsilon, T).$$

$\square$

Corollary 2 (Shadowing) Given $u(t)$.

There exists

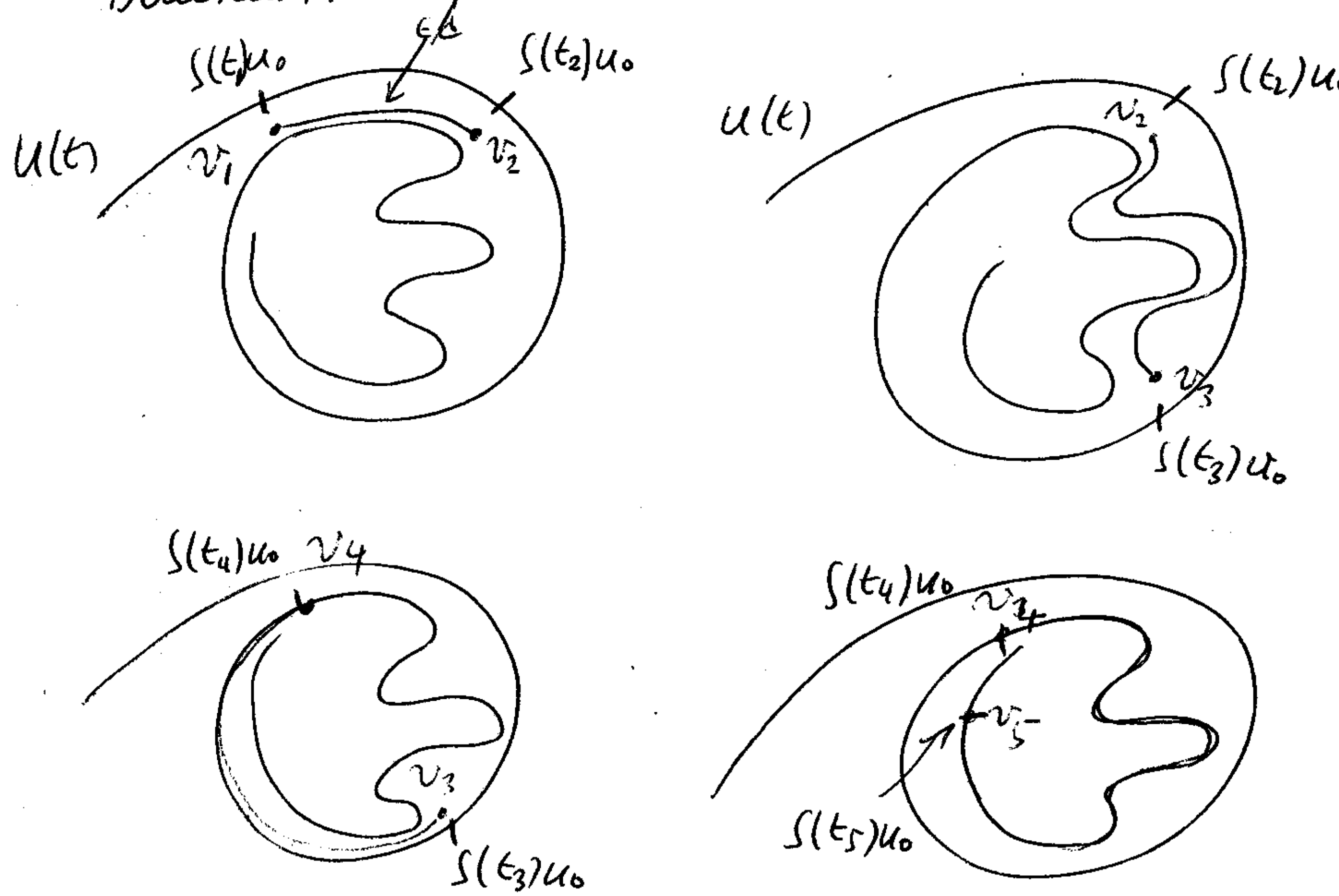
- a sequence of errors $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$
- an increasing sequence of times $\{t_n\}_{n\in\mathbb{N}}$, $t_{n+1} - t_n \to \infty$
- a sequence of points $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$

such that

$$|u(t) - S(t-t_n)\, v_n| \leq \varepsilon_n \quad , \quad t_n \leq t \leq t_{n+1}.$$

The jumps satisfy $\quad |v_{n+1} - S(t_{n+1} - t_n)\, v_n| \to 0.$

Proof. Follows directly from repeated application of Theorem 1:



etc.