(4.2) Structure of the Attractor

<u>Theorem 1</u>: All complete bounded orbits lie in $\mathcal{A}$.
If $S(t)$ is injective, then $\mathcal{A}$ is the union of all complete orbits.

A <u>complete</u> orbit $u(t)$ exists <u>for all $t \in \mathbb{R}$</u>

<u>Proof</u> If $\mathcal{O}$ is a complete bounded orbit, it is invariant and it is attracted by $\mathcal{A}$:

$$\operatorname{dist}(S(t)\mathcal{O}, \mathcal{A}) < \varepsilon \quad \text{for all } t \geq t_0(\mathcal{O}, \varepsilon).$$

If $\mathcal{O} \not\subset \mathcal{A}$ then there exists $x \in \mathcal{O}$: $\operatorname{dist}(x, \mathcal{A}) > \varepsilon$.
Since $\mathcal{O}$ is complete, there is a $z \in \mathcal{O}$: $S(t)z = x$, $t > t_0(\mathcal{O}, \varepsilon)$. Then

$$\varepsilon < \operatorname{dist}(S(t)z, \mathcal{A}) < \varepsilon \qquad \lightning$$

If $S(t)$ is injective, then for $x \in \mathcal{A}$ the complete orbit of x is in $\mathcal{A}$ (Theorem 4 in (4.1)).

$\square$

<u>Definition</u> stable / unstable manifold:

$$W^s(z) = \{u_0 \in H : \ S(t)u_0 \to z \text{ as } t \to \infty\}$$

$$W^u(z) = \{u_0 \in H : \ S(t)u_0 \text{ defined for all } t \in \mathbb{R}, \ S(-t)u_0 \to z \text{ as } t \to \infty \}$$

$$W^u(X) = \{u_0 \in H : \ S(t)u_0 \text{ defined for all } t \in \mathbb{R}, \ \operatorname{dist}(S(-t)u_0, X) \to 0 \}$$

Theorem 2 X compact invariant set then

$$W^u(X) \subset \mathcal{A}$$

Proof: $u \in W^u(X) \Rightarrow u$ lies on a complete orbit,

for $t \to -\infty$: $\mathrm{dist}\,(S(t)u, X) \to 0$

for $t \to +\infty$: $\mathrm{dist}\,(S(t)u, \mathcal{A}) \to 0$

$\Rightarrow \{u(t)\}$ is complete and bounded,

hence lies in $\mathcal{A}$ by Theorem 1.

$\square$

Previous example: