§5 Global Attractors

H Banach space, often Hilbert-space

$\{H, \{S(t)\}_{t\geq 0}\}$ a <u>semidynamical system</u>

(i) $S(0) = I, \quad S(t+s) = S(t)\,S(s)$

(ii) $S(t)u_0 = u(t)$ is continuous in both t and u_0.

We call $S(t)$ a C^0-semigroup.

It follows the <u>uniform continuity on compact sets</u> (see Exercise 10.1): For each $K \subset\subset H$ there exists a nondecreasing function $\delta_K(t, u)$ with

$$\delta_K(t, 0) = 0, \quad \delta_K(0, d) = d \quad \text{and}$$

$$|S(t)u_0 - S(t)v_0| \leq \delta_K\big(T, |u_0 - v_0|\big)$$

<u>Now</u> we use $|\cdot| = \|\cdot\|_H, \qquad \|\cdot\| = \|\cdot\|_V$

(5.1) Dissipation, Limit Sets and Attractors

<u>Definition 1</u>

a) A bounded set $B \subset H$ is attracting, if for each $x_0 \in H$ there is a $t_0(x_0)$ such that

$$S(t)x_0 \in B \quad \text{for all } t > t_0(x_0).$$

b) $S(t)$ is <u>point-dissipative</u> if H has an attracting set B.

c) A bounded set $B \subset H$ is <u>absorbing</u> if it is attracting, and for each bounded set $X \subset H$ there is a time $t_1(X) > 0$ such that

$$S(t)X \subset B \quad \text{for all} \quad t \geq t_1(X).$$

d) $S(t)$ is <u>bounded dissipative</u>, if H has an absorbing subset.

e) $S(t)$ is <u>dissipative</u>, if H has a <u>compact</u> absorbing subset.

f) $X \subset H$ is <u>positively invariant</u>, if $S(t)X \subset X$, $t \geq 0$

g) $X \subset H$ is <u>invariant</u>, if $S(t)X = X$, $t \geq 0$.

<u>Lemma 1</u>: $H = \mathbb{R}^n$.

(i) $S(t)$ point dissipative $\iff$ $S(t)$ dissipative

(ii) B attracting set, then for each $\varepsilon > 0$ the set

$$B_\varepsilon = \overline{\bigcup_{0 \leq t < \infty} S(t)\left(\overline{N}(B; \varepsilon)\right)}$$

is absorbing.

<u>Proof</u>: <u>clear</u>: dissipative $\Rightarrow$ point dissipative.

Now assume $S(t)$ is point dissipative with attracting set B.

B_ε is bounded and closed (by construction) hence compact.

Let $X \subset H$ be bounded. For each $z \in X$ there is a $t_0(z) > 0$ such that $S(t)x \in B$ for all $t \geq t_0(z)$. Since the flow is continuous there exists a neighborhood $N(z)$ such that $S(t)N \subseteq N(B; \varepsilon)$ for all $t \geq t_0(z)$. $X \subset \mathbb{R}^n$ bounded, hence $\overline{X}$ is compact. We take a finite subcover $\{N(z_i)\}_{i=1,\dots,m}$ and we choose $t^* := \max_{i=1,\dots,m} t_0(z_i)$.

Then $S(t)X \subset B_\varepsilon$ for all $t \geq t^*$.

$\square$

<u>Definition 2</u> <u>ω-Limit sets</u> of sets $X \subset H$:

$$\omega(X) = \{y : \text{exists } t_n \to \infty,\ x_n \in X : S(t_n)x_n \to y\}$$

$$= \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)X}$$

<u>Lemma 2</u>: $X \subset H$. If for some $t_0 > 0$ the set $\overline{\bigcup_{t \geq t_0} S(t)X}$ is compact, then $\omega(X)$ is nonempty, compact and invariant.

Proof: For $t \geq t_0$ the sets $W_t := \overline{\bigcup_{s \geq t} S(s)X}$ are nonempty, compact and nested: $W_t \subset W_\vartheta$ for $t \geq \vartheta$. Hence their intersection $\omega(X) = \bigcap_{t \geq 0} W_t$ is nonempty and compact.

$S(t)\omega(X) \subset \omega(X)$: Let $y \in \omega(X)$, $\lim_{n\to\infty} S(t_n)x_n = y$ then $S(t)y = \lim_{n\to\infty} S(t+t_n)x_n = \lim_{n\to\infty} S(\sigma_n)x$, $\sigma_n = t+t_n$

$\in \omega(X)$.

$\omega(X) \subset S(t)\omega(X)$: Let $y \in \omega(X)$, $\lim_{n\to\infty} S(t_n)x_n = y$. Study $t_n > t_0 + t$, then

$$y = \lim_{n\to\infty} S(t_n)x_n = \lim_{n\to\infty} S(t)S(t_n - t)x_n$$
$$= S(t) \lim_{n\to\infty} S(t_n - t)x_n$$

Now $S(t_n - t)x_n$ converges, since $t_n - t > t_0$.

$\square$

Suppose $S(t)$ is dissipative with absorbing set B. Is $\omega(B)$ a good candidate for an Attractor?

<u>Definition</u> The <u>global attractor $\mathcal{A}$</u> is the maximal compact invariant set $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ and the minimal set that that <u>attracts all bounded sets</u>, i.e.

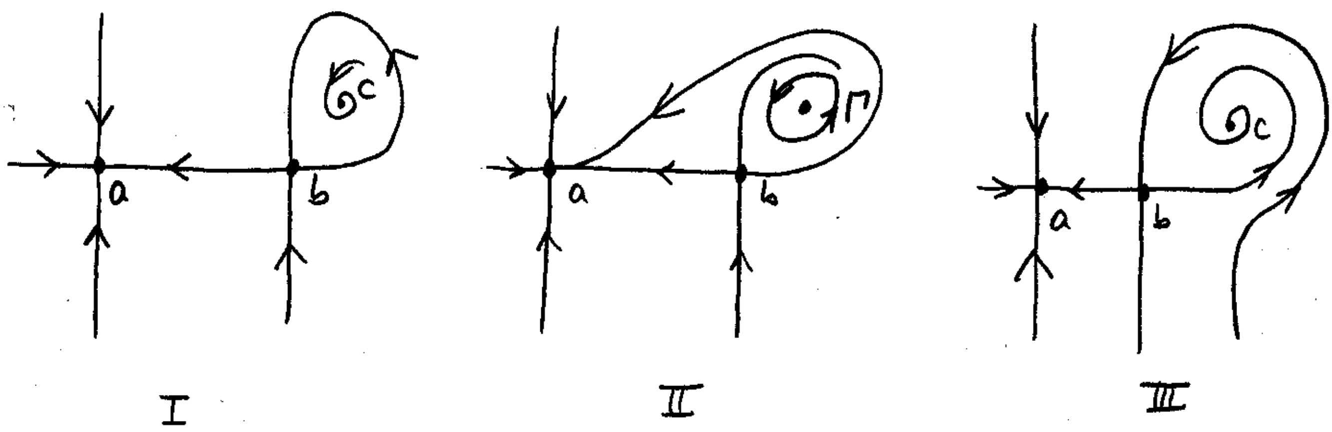
$$\operatorname{dist}\left(S(t)X, \mathcal{A}\right) \longrightarrow 0 \quad \text{as} \quad t \to \infty$$

for all $X \subset H$ bounded.

<u>Q</u>: How to find $\mathcal{A}$:

<u>Candidate 1</u> $\quad A := \bigcup_{x \in B} \omega(x)$

<u>Examples</u>



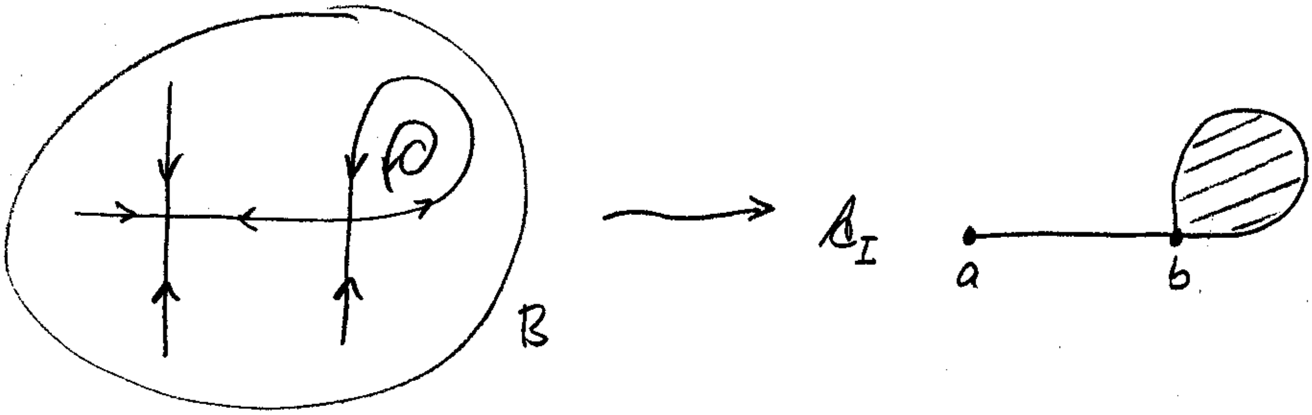
$A_{I} = \{a, b, c\}$, $\quad A_{II} = \{a, b, c, \Gamma\}$, $\quad A_{III} = \{a, b, c\}$

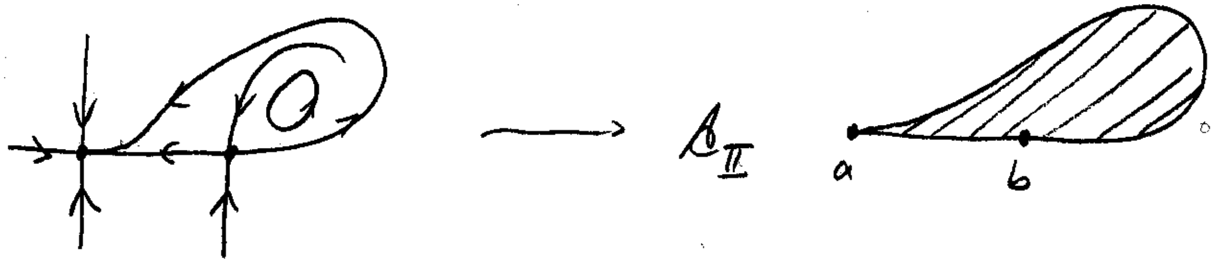
"explosion" of the attractor.

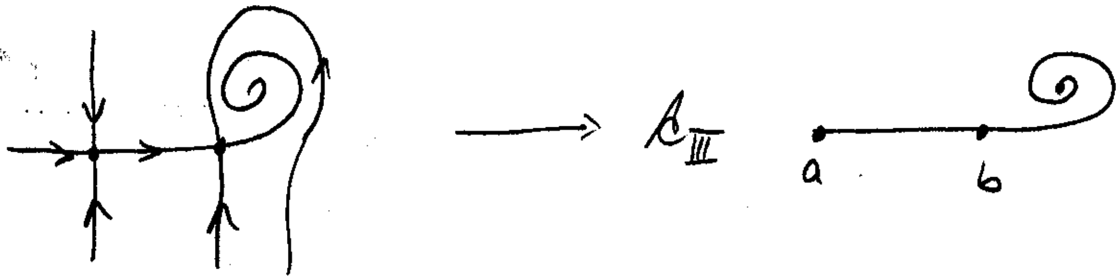
But $A_I = \{a, b, c\}$ is not maximal invariant, because

$$\tilde{A}_I = \{ a \longleftarrow b \quad \cdot c \}$$ is also invariant.

<u>Candidate 2</u> $\mathcal{A} = \omega(B)$.







<u>Theorem 1</u> $S(t)$ dissipative, B absorbing, then $\mathcal{A} = \omega(B)$ is the global attractor.

If H is connected, then $\mathcal{A}$ is connected.

Proof: (i) $\omega(B)$ is nonempty, compact and invariant (Lemma 2 in (5.1)).

(ii) $\omega(B)$ is the maximal compact invariant set. If Y is also invariant, then $S(t)Y = Y$, $\omega(Y) = Y$ and $S(t)Y \subset B$ for all $t \geq t_0(Y)$. Then $\omega(Y) \subset \omega(B)$.

(iii) To show that $\mathcal{A}$ attracts bounded sets we assume that there exists a bounded set X such that $\text{dist}(S(t_n)X, \mathcal{A}) \geq \delta$ for $t_n \to \infty$.

There are points x_n: $\text{dist}(S(t_n)x_n, \mathcal{A}) \geq \frac{\delta}{2}$

For t large enough $S(t_n)x_n \in B$ and B is cp.

Hence we find a convergent subsequence $S(t_{n_j})x_{n_j} \to \beta \in B$, and still $\text{dist}(\beta, \mathcal{A}) \geq \frac{\delta}{2}$.

$$\beta = \lim_{j\to\infty} S(t_{n_j})x_{n_j} = \lim_{j\to\infty} S(t_{n_j} - t_0(X))\, S(t_0(X))x_{n_j}$$

$$\subset \omega(B) \qquad ↯ \text{ contradiction.}$$

(iv) $\mathcal{A}$ is the minimal set that attracts bounded sets and it is invariant.

Indeed, if there is a smaller set Γ then this would attract $\omega(B)$, hence $\omega(B) \subset \Gamma$. Since $\omega(B)$ is invariant we get $S(t)\mathcal{A} = \mathcal{A}$.

(v) $\mathcal{A}$ connected:

Suppose $\mathcal{A}$ is not connected. Then exists $\mathcal{O}_1, \mathcal{O}_2$ open
$\mathcal{A} \subset \mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{A} \cap \mathcal{O}_1 \neq \emptyset$, $\mathcal{A} \cap \mathcal{O}_2 \neq \emptyset$, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$.
Let $\mathcal{U}$ be a ball in H that covers $\mathcal{B}$.
$\mathcal{U}$ is connected and $\omega(\mathcal{U}) = \omega(\mathcal{B})$. Since $S(t)$ is continuous $S(t)\mathcal{U}$ is connected. Hence for t large enough $S(t) \cap \mathcal{O}_i \neq \emptyset$ for all $i = 1,2$.
But $S(t)\mathcal{U} \not\subset \mathcal{O}_1 \cup \mathcal{O}_2$. Then for each $t_n \to \infty$ we find $x_n \in \mathcal{U}$, $S(t_n)x_n \notin \mathcal{O}_1 \cup \mathcal{O}_2$.
For $t_n > t_0(\mathcal{U})$ the sequence $S(t_n)x_n$ converges in $\mathcal{B}$ to $x_0 \notin \mathcal{O}_1 \cup \mathcal{O}_2$, but $x_0 \in \mathcal{A}$ ↯

□

<u>Definition</u>: A semigroup is <u>injective</u> on $\mathcal{A}$, if $S(t)u_0 = S(t)v_0 \in \mathcal{A}$ for some $t \geq 0$, then $u_0 = v_0$.

<u>Theorem 2</u> If a semigroup $S(t)$ is injective on $\mathcal{A}$ then each trajectory in $\mathcal{A}$ is <u>global</u> (i.e. defined for $t \in \mathbb{R}$). We have $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$ and $\left(\mathcal{A}, \{S(t)\}_{t \in \mathbb{R}}\right)$ is a <u>dynamical system</u>

Proof: For $u \in \omega(B)$ we know that $S(t)u \in \omega(B)$. Since $S(t)$ is injective we find a unique $v \in \omega(B)$ with $u = S(t)v$. We define $S(-t)u = v$. By this definition we have $S(0) = I$, $S(t+s) = S(t)S(s)$ for all $s,t \in \mathbb{R}$ on $\mathcal{A}$.

To show that $S(t)$ is continuous for $t<0$, we look at a sequence $u_n \to u$ in $\omega(B)$. By the above definition we find $v_n = S(-t)u_n$ and $v = S(-t)u$, $v_n, v \in \omega(B)$ hence $\{v_n\}$ has a convergent subsequence $v_{n_j} \to z$ with

$$S(t)z = S(t)\lim_j v_{n_j} = S(t)\lim_j S(-t)u_{n_j}$$

$$= \lim_j S(t-t)u_{n_j} = \lim_j u_{n_j} = u$$

Since $S(t)$ is injective we must have $z = v$, hence $S(-t)u_n \longrightarrow v = S(-t)u$

$\square$.

<u>Example: Lorenz equations</u>

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \mu x - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$$

We showed a positively invariant absorbing set

$E = \{(x,y,z);\ V(x,y,z) \leq C + \varepsilon\}$

$V(x,y,z) = \mu x^2 + \sigma y^2 + \sigma(z-2\mu)^2$.

$$\Rightarrow \quad \mathcal{A} = \omega(E) \neq \emptyset.$$