(4.4) Weak Solutions

Def: A family of functions $u(t) \in H$ is <u>weakly continuous</u> in t_0, if

$$\lim_{t \to t_0} (u(t) - u(t_0), \phi) = 0 \quad \text{for all } \phi \in H.$$

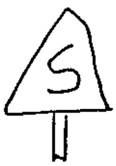
Theorem Let $f \in L^2_{loc}(0,T; V^*)$, $u_0 \in H$, then there exists a weak solution $u(t)$ to

$$\frac{du}{dt} + \nu A u + B(u,u) = f \qquad (1)$$

such that

$$u \in L^\infty(0,T; H) \cap L^2(0,T; V).$$

(*) holds a equality in $L^p(0,T; V^*)$ with $p=2$ for $n=2$ and $p = \frac{4}{3}$ for $n=3$.
The solution is weakly continuous in H.



: No uniqueness - statement!

Proof: (i) <u>Galerkin approximation</u> $\quad u_n = \sum_{j=1}^{n} u_{n_j} \cdot w_j$

$$\frac{du_n}{dt} + \nu A u_n + P_n B(u_n, u_n) = P_n f \qquad (2)$$

Multiply (2) by u_n and integrate

$$\frac{1}{2} \frac{d}{dt} \|u_n\|^2_{L^2} + \nu (Au_n, u_n) + (P_n B(u_n, u_n), u_n) = (P_n f, u_n)$$

$$(P_n B(u_n,u_n), u_n) = (B(u_n,u_n), P u_n) = b(u_n,u_n,u_n) = 0$$

(Prop. 1 in (4.3)).

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \nu\|u_n\|_V^2 = \langle f, u_n\rangle$$

$$\leq \|f\|_{V^*}\|u_n\|_V$$

$$\overset{\text{Young}}{\leq} \frac{\nu}{2}\|u_n\|_V^2 + \frac{\|f\|_{V^*}^2}{2\nu}$$

$$\Longrightarrow \quad \frac{d}{dt}\|u_n\|_{L^2}^2 + \nu\|u_n\|_V^2 \leq \frac{\|f\|_{V^*}^2}{\nu}$$

Integrate from 0 to t:

$$\|u_n(t)\|_{L^2}^2 + \nu\int_0^t \|u_n\|_V^2\,dt \leq \|u_n(0)\|_{L^2}^2 + \frac{\|f\|_{L^2(0,T;V^*)}^2}{\nu}$$

$\Longrightarrow$ $\{u_n\}$ uniformly bounded in $L^\infty(0,T;H)$

$\{u_n\}$ —"— $L^2(0,T;V)$

Alaoglu: $u_n \xrightarrow{*} u$ in $L^\infty(0,T;H)$

reflexive compactness: $u_n \rightharpoonup u$ in $L^2(0,T;V)$

(iii) $B(u_n,u_n)$

$$\frac{du_n}{dt} = -\nu \underbrace{Au_n}_{\in L^2(0,T;V^*)} - \underbrace{P_n B(u_n,u_n)}_{?} + \underbrace{P_n f}_{\in L^2(0,T;V^*)}$$

Proposition 2 in (4.3) $\Rightarrow$

$$\|B(u,u)\|_{V^*} \le k \begin{cases} \|u\|_{L^2} \; \|u\|_{H^1} & , n=2 \\ \|u\|_{L^2}^{1/2} \; \|u\|_{H^1}^{3/2} & , n=3 \end{cases}$$

$\underline{n=2}$:
$$\|P_n B(u_n,u_n)\|^2_{L^2(0,T;V^*)} \le k \int_0^T \|u_n\|^2_{L^2} \; \|u_n\|^2_{H^1} \, ds$$
$$\le k \, \|u_n\|^2_{L^\infty(0,T;H)} \; \|u_n\|^2_{L^2(0,T;V)}$$

$\underline{n=3}$:
$$\|P_n B(u_n,u_n)\|^2_{L^{4/3}(0,T;V^*)} \le k \int_0^T \|u_n\|^{2/3}_{L^2} \; \|u_n\|^2_{H^1} \, ds$$
$$\le k \, \|u_n\|^{2/3}_{L^\infty(0,T;H)} \; \|u_n\|^2_{L^2(0,T;V)}$$

$\Rightarrow$ $P_n B(u_n,u_n)$ and also $\frac{du_n}{dt}$ is uniformly bounded in $L^2(0,T;V^*)$ for $n=2$
$L^{4/3}(0,T;V^*)$ for $n=3$.

Compactness theorem 3 in (2.8) [R. Thm 8.1]
$\Rightarrow$ $u_n \longrightarrow u$ in $L^2(0,T;H)$ (<u>strong convergence</u>)

(iii) As before in R-D-eq: $\frac{du_n}{dt} \xrightarrow{*} \frac{du}{dt}$ in $L^p(0,T;V^*)$
$p=2$ for $n=2$, $p=\frac{4}{3}$ for $n=3$.

(vi) <u>Claim:</u> $B(u_n,u_n) \xrightarrow{*} B(u,u)$ in $L^p(0,T;V^*)$

Choose $w \in L^q(0,T;V)$

$$\int_0^T \langle B(u_n,u_n), w\rangle\, dt = \int_0^T b(u_n,u_n,w)\, dt = -\int_0^T b(u_n,w,u_n)\, dt$$

$$= -\int_0^T \sum_{i,j=1}^n \int_Q (u_n)_i\,(D_i w_j)\,(u_n)_j\, dx\,dt$$

If we study $\int_0^T b(u_n,u_n,w) - b(u,u,w)\, dt$

we find terms of the form:

$$\int_0^T \int_Q (u_{n_i} - u_i)(D_i w_j)\, u_j\, dx\,dt$$

$$\le \underbrace{\| u_{n_i} - u_i \|_{L^2(0,T;H)}}_{\downarrow\ 0 \text{ in } L^2} \ \underbrace{\| D_i w_j \|_{L^2(0,T;H)}}_{\text{bounded, since } w_i \in L^q(0,T;V)} \ \underbrace{\| u_j \|_{L^\infty(0,T;H)}}_{\uparrow\ \text{bounded}}$$

Then $B(u_n,u_n) \xrightarrow{*} B(u,u)$ and also

$P_n B(u_n,u_n) \xrightarrow{*} B(u,u)$ in $L^p(0,T;V^*)$.

Hence indeed: $\frac{du}{dt} + \nu A u + B(u,u) = f$ in $L^p(0,T; V^*)$.

(v) $u(0) = u_0$:

Multiply $\left\{ \begin{matrix} N\text{-}S\text{-eq} \\ \text{Galerkin App.} \end{matrix} \right\}$ by $\phi \in C^1([0,T], V)$, $\phi(T) = 0$,

integrate, take the weak $*$ limit and substract the equations:

$$\lim_{n\to\infty} \big(u_n(0), \phi(0) \big) - \big(u(0), \phi(0) \big) = 0.$$

(vi) weak continuity into H

Apply (1) to $v \in V$:

$$\left(\frac{du}{dt}, v\right) + \nu (Au, v) + b(u,u,v) = \langle f, v\rangle$$

integrate from t_0 to t:

$$(u(t) - u(t_0), v) = -\nu \int_{t_0}^{t} \underbrace{(Au, v)}_{\substack{\in L^1(0,T;\mathbb{R}) \\ \text{since } u \in L^2(0,T;V)}} ds + \int_{t_0}^{t} \underbrace{b(u(s),u(s),v)}_{\substack{\in L^1(0,T;\mathbb{R}) \\ \text{by Prop. 2 in (4.3)}}} ds$$

$$+ \int_{t_0}^{t} \underbrace{\langle f(s), v\rangle}_{\substack{\in L^1(0,T;\mathbb{R}) \text{ since } f \in L^2_{loc}(0,T;V^*)}} ds \longrightarrow 0 \quad \text{as } t \to t_0$$

Since $V \subset H$ dense

$(u(t) - u(t_0), \phi) \to 0$ as $t \to t_0$ for all $\phi \in H$.

$\square$