(4.3) Weak formulation of the N-S-eq.

N-S-eq: $u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad , \nabla \cdot u = 0$

multiply with $v \in V = \{ \dot{H}^1(\Omega) : \nabla \cdot u = 0 \}$ and integrate

$$\left(\frac{\partial u}{\partial t}, v\right) \underbrace{- \nu (\Delta u, v)}_{= +\nu a(u,v)} + ((u \cdot \nabla) u, v) + \underbrace{(\nabla p, v)}_{= -(p, \underbrace{\nabla \cdot v}_{=0}) = 0} = (f, v)$$

Define a tri-linear form b:

$$b(u, v, w) = ((u \cdot \nabla) v, w)$$

Then for each $v \in V$:

$$\underbrace{\left(\frac{\partial u}{\partial t}, v\right)}_{\in V^*} + \underbrace{\nu a(u,v)}_{= \nu \langle Au, v \rangle,\ Au \in V^*} + b(u,u,v) = \underbrace{\langle f, v \rangle}_{\in V^*}$$

We define a linear form on V, $B(u,u)$ by:

$$\langle B(u,u), v \rangle = b(u,u,v) \quad , \quad B(u,u) \in V^*.$$

Weak formulation:

$$\boxed{\frac{du}{dt} + \nu A u + B(u,u) = f \quad \text{in } V^*}$$

Note: No pressure term.

Properties of the trilinear form $b(u,v,w)$:

Proposition 1: $n=2,3$ $\quad u \in H = \{ \mathbb{L}^2, \nabla\cdot u = 0\}$, $v,w \in V$:

(i) $b(u,v,w) = -b(u,w,v)$

$\quad b(u,v,v) = 0$

(ii) $n=2$ + periodic b.c.

$\quad b(u,u,Au) = 0 \qquad \forall\, u \in \mathcal{D}(A)$

(iii) $b(v,u,Au) + b(u,v,Au) + b(u,u,Av) = 0 \qquad u,v \in \mathcal{D}(A)$

Proof. (i) Integration by parts

(ii) $b(u,u,Au) = ((u\cdot\nabla)u, Au)$

periodic b.c. $= -((u\cdot\nabla)u, \Delta u)$

$$= -\sum_{i,j,k} \left((u_i \cdot \partial_i) u_j, \partial_k^2 u_j \right)$$

$$= \sum_{i,j,k} \left(\partial_k (u_i \partial_i) u_j, \partial_k u_j \right)$$

$$= \sum_{i,j,k} \left(\partial_k u_i \, \partial_i u_j, \partial_k u_j \right) + \underbrace{\sum_{i,j,k} \left(u_i \partial_i \, \partial_k u_j, \partial_k u_j \right)}_{= -\left(\nabla\cdot u, \frac{1}{2}(\partial_k u_j)^2\right) = 0}$$

write down componentwise

$$\underbrace{(\partial_1 u_1 + \partial_2 u_2)}_{= \nabla\cdot u = 0}(\cdots)$$

(iii) Use (ii) with $w = u + \varepsilon v$ and compare $O(\varepsilon)$-terms.

Proposition 2 $m = 2, 3$:

(i) $u \in \mathbb{L}^\infty$, $v \in V$, $w \in H$

$$|b(u,v,w)| \leq \|u\|_\infty \|v\|_V \|w\|_H$$

(ii) $u, v, w \in V$:

$$|b(u,v,w)| \leq k \begin{cases} \|u\|_{L^2}^{1/2} \|u\|_{H^1}^{1/2} \|v\|_{H^1} \|w\|_{L^2}^{1/2} \|w\|_{H^1}^{1/2}, & m=2 \\ \|u\|_{L^2}^{1/4} \|u\|_{H^1}^{3/4} \|v\|_{H^1} \|w\|_{L^2}^{1/4} \|w\|_{H^1}^{3/4}, & m=3 \end{cases}$$

(iii) $u \in V$, $v \in \mathcal{D}(A)$, $w \in H$:

$$|b(u,v,w)| \leq k \begin{cases} \|u\|_{L^2}^{1/2} \|u\|_{H^1}^{1/2} \|v\|_{H^1}^{1/2} \|Av\|_{L^2}^{1/2} \|w\|_{L^2} & m=2 \\ \|u\|_{H^1} \|v\|_{H^1}^{1/2} \|Av\|_{L^2}^{1/2} \|w\|_{L^2} & m=3 \end{cases}$$

Proof: Hölders inequality and Sobolev embedding (R. p. 243 ff.).