§4 The Navier-Stokes equations

(4.1) Preassure and Fluid Velocity

Model for fluid flow.

incompressible fluid (= constant density, $\rho = 1$)

$u(x,t) \in \mathbb{R}^n$ velocity, $p(x,t)$: preassure

N-S-eq
$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(x,t) & \text{Newton's law} \\ \nabla \cdot u = 0 & \text{conservation of mass.} \end{cases}$$

$\nu > 0$: kinematic viscosity

on $Q = [0, L]^n$, $n = 2, 3$

+ periodic boundary conditions $u(x + Le_j, t) = u(x,t)$.

Define Sobolev spaces with periodic boundary conditions and mean zero:

$$\dot{H}^s_p(Q) = \left\{ u \in H^s(Q) : u(x + Le_j) = u(x), \int_Q u(x)\,dx = 0 \right\}$$

Poincaré - inequality (R. lemma 5.40)

$$\|u\|_{L^2} \le \left(\frac{L}{2\pi}\right) \|Du\|_{L^2}$$

$\lambda_1 := \frac{L^2}{4\pi^2}$ first $\neq 0$ e-value of $-\Delta$ on Q with periodic boundary conditions.

We assume $\int_Q u_0(x)\,dx = 0$, $\int_Q f(x,t)\,dx = 0$

Then

<u>Lemma 1</u> $\int_Q u(x,t)\,dx = 0$ for all $t \geq 0$.

<u>Proof</u>:

$$\frac{d}{dt}\int_Q u(x,t)\,dx = \int_Q \left(\nu\Delta u - (u\cdot\nabla)u - \nabla p + f\right)dx$$

$$= \nu\int_Q \Delta u - \int_Q (u\cdot\nabla)u - \int_Q \nabla p + \int_Q f\,dx$$

$\int_Q f\,dx = 0$ from assumption

$$\int_Q \nabla p\,dx = \int_Q \left(\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}\right)^T dx$$

$$\int_Q \frac{\partial p}{\partial x_1}\,dx = \int_0^L \cdots \int_0^L \frac{\partial p}{\partial x_1}\,dx_1\,dx_2 \ldots dx_n$$

$$= \int_0^L \cdots \int_0^L \Big(\underbrace{p(L, x_2, \ldots, x_n) - p(0, x_2, \ldots, x_n)}_{=0\text{, since } p \text{ periodic}}\Big)dx_1 \ldots dx_n$$

$$\int_Q \Delta u \, dx = \int_0^L \cdots \int_0^L \frac{\partial^2}{\partial x_1^2} u + \cdots + \frac{\partial^2}{\partial x_n^2} u \, dx_1 \cdots dx_n$$

$$\int_0^L \cdots \int_0^L \frac{\partial^2}{\partial x_1^2} u \, dx_1 \cdots dx_n = \int_0^L \cdots \int_0^L \underbrace{\frac{\partial u}{\partial x_1}(L, \ldots) - \frac{\partial u}{\partial x_1}(0, \ldots)}_{=0, \ u \text{ periodic}} dx_2 \cdots dx_n.$$

$$\int_Q (u \cdot \nabla) u \, dx = \sum_{j=1}^n \int_Q \left(u_j \frac{\partial}{\partial x_j} \right) u \, dx = - \sum_{j=1}^n \int_Q \left(\frac{\partial}{\partial x_j} \cdot u_j \right) u \, dx$$

$$= - \int_Q (\nabla \cdot u) u \, dx = 0 \quad \text{since } \nabla \cdot u = 0.$$

$$\Rightarrow \int_Q u(x,t) \, dx = 0 \quad \text{for all } t \geq 0. \qquad \square$$

Hence we work in $\dot{L}^2(Q)$.