(2.8) Banach-space valued functions [R. §7.1]

Reaction diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u)$$

solutions: $u(x,t)$ for each t: $[u(t)](x)$, $u(t) \in X$.

$I := (0,T)$ or $[0,T]$:

$C^0(I,X) = \{u : I \to X \text{ continuous in } t\}$

$C^0(0,T; L^p(\Omega))$ has norm

$$\|u\|_{C^0(0,T;L^p(\Omega))} = \sup_{0\le t\le T} \|u(t)\|_{L^p}$$

$L^p(0,T; L^q(\Omega))$ has norm

$$\|u\|_{L^p(0,T;L^q(\Omega))} = \left(\int_0^T \|u(t)\|^p_{L^q(\Omega)}\,dt\right)^{\frac{1}{p}}$$

$$= \left(\int_0^T \left(\int_\Omega |u(x,t)|^q\,dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}}$$

$$\|u\|_{L^p(0,T;L^p(\Omega))} = \left(\int_0^T \int_\Omega |u(x,t)|^p\,dx\,dt\right)^{\frac{1}{p}}$$

$$= \|u\|_{L^p((0,T)\times\Omega)}.$$

$L^2(0,T,L^2(\Omega))$ has inner product $\displaystyle\int_0^T\int_\Omega u(x,t)v(x,t)\,dx\,dt$

Proposition 1: $u \in W^{1,p}(0,T;X)$ $1 \le p \le \infty$. Then

(i) $u(t) = u(s) + \int_s^t \frac{du}{dt}(\tau)\,d\tau$ for each $0 \le s \le t \le T$.

(ii) $u \in C^0(0,T;X)$

(iii) $$\sup_{0 \le t \le T} \|u(t)\|_X \le C \|u\|_{W^{1,p}(0,T;X)}.$$

Proof: In class presentation No. 4.

Assume now that $u \in L^2(0,T;H^1(\Omega))$, $\frac{du}{dt} \in L^2(0,T;H^{-1}(\Omega))$

Since $H^1 \subset H^{-1}$ it follows

$$u \in H^1(0,T;H^{-1}(\Omega)).$$

Prop 1 $\Rightarrow$ $u \in C^0(0,T;H^{-1}(\Omega))$

Theorem 2: $u \in C^0(0,T;L^2(\Omega))$.

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<u>Theorem 3</u>: Let $X \subset\subset H \subset Y$ B-spaces and H reflexive.

Suppose $\{u_n\} \subseteq L^2(0,T;X)$ uniformly bounded.

$\left\{\frac{\partial u_n}{\partial t}\right\} \subseteq L^p(0,T;Y)$ uniformly bounded, for some $p>1$

Then there is a subsequence that converges strongly in $L^2(0,T;H)$.

<u>Example</u>: Solution of $u_{nt} = \Delta u_n + f_n(t)$:

$u_n \subset L^2(0,T, H_0^1)$, $\frac{\partial}{\partial t} u_n \subseteq L^2(0,T; H^{-1})$

$X = H_0^1$, $H = L^2(\Omega)$, $Y = H^{-1}(\Omega)$.

If $\{u_n\}$, $\left\{\frac{\partial u_n}{\partial t}\right\}$ uniformly bdd, then we have a convergent subsequence in $L^2(0,T; L^2(\Omega)) = L^2((0,T)\times\Omega)$.