## (2.5) Linear Operators

a) $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ B-spaces.

$A: X \to Y$

$A$ linear $\iff$ $A(\alpha x + \beta y) = \alpha A x + \beta A y$

bounded $\iff$ $\|Ax\|_Y \le M \|x\|_X \quad \forall x \in X$

$\mathcal{L}(X,Y) := \{ A: X \to Y, \text{ linear + bounded} \}$.

$\mathcal{L}(X,Y)$ is a B-space with the operator-norm:

$$\|A\|_{op} := \inf\{ M : \|Ax\|_Y \le M \|x\|_X \ \forall x \in X \}$$

$$= \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

$$= \sup_{\|x\|_X = 1} \|Ax\|_Y$$

Proposition 1: Let $A: X \to Y$ be a linear map.
$A$ bounded $\iff$ $A$ continuous.

Proof:

„$\Rightarrow$": $A$ bounded $\Rightarrow$ $\|A(x_n - x)\|_Y \le \|A\|_{op} \|x_n - x\|_X$

$\Rightarrow$ $A$ continuous.

„$\Leftarrow$": $A$ continuous, assume $A$ not bounded. Then for each $n$ there exists $y_n \in X$: $\|Ay_n\|_Y \ge n^2 \|y_n\|_X$

Then $x_n := \dfrac{y_n}{n\|y_n\|_X} \longrightarrow 0$ but $\|Ax_n\|_Y \ge n$ $\lightning$

Example 1 The integral operator

$$Ku(x) = \int_\Omega k(x,y)\, u(y)\, dy \quad \text{with} \quad \iint_{\Omega\Omega} |k(x,y)|^2 dx\,dy = C < \infty$$

is bounded on $L^2(\Omega)$.

Proof:
$$\|Ku\|_2^2 = \int_\Omega \left| \int_\Omega k(x,y)u(y)\,dy \right|^2 dx$$

$$\overset{C.S}{\leq} \int_\Omega \left( \int_\Omega |k(x,y)|^2 dy \right)\left( \int_\Omega |u(y)|^2 dy \right) dx$$

$$= C \, \|u\|_2^2$$

qed

Example 2: $A := -\frac{d^2}{dx^2}$ on $L^2[0,1]$. Since $C^2[0,1]$ is dense on $L^2[0,1]$ we define $A$ only on its domain $\mathcal{D}(A) = \{ f \in C^2[0,1] : Af \in L^2[0,1] \}$.

$A : \mathcal{D}(A) \to L^2[0,1]$. $A$ is unbounded.

Proof.
$$\|A e^{-kx}\|_2 = \left\| -\frac{d^2}{dx^2} e^{-kx} \right\|_2 = k^2 \|e^{-kx}\|_2$$

qed

b) For an unbounded operator the domain belongs to the operator. Write $(A, \mathcal{D}(A))$.

e.g. $\left( -\frac{d^2}{dx^2}, \mathcal{D}(A) \text{ from above} \right)$

or $\left( -\frac{d^2}{dx^2}, \mathcal{D}(A) = \{ f \in C^2[0,1],\ Af \in L^2[0,1],\ f(0)=0,\ f(1)=0 \} \right)$, ...

or $\left( A = -\frac{d^2}{dx^2}, \quad \mathcal{D}(A) = \left\{ f \in C^2[0,1],\ Af \in L^2[0,1], \ \frac{\partial f}{\partial x}(0) = 0, \quad \frac{\partial f}{\partial x}(1) = 0 \quad \right\} \right)$

b) <u>Range</u> : $A : \mathcal{D}(A) \to Y \qquad R(A) := \{ g \in Y : g = Af, \ f \in \mathcal{D}(A) \}$

<u>Kernel</u> $\quad \operatorname{Ker} A = \{ f \in \mathcal{D}(A) : Af = 0 \}$.

<u>Lemma</u> $\quad A : \mathcal{D}(A) \to R(A)$ invertible, iff $\operatorname{Ker} A = \{0\}$.

c) <u>Definition</u> : $K : X \to Y$ is <u>compact</u>, if for each $\mathcal{U} \subset X$ bounded, $\overline{K(\mathcal{U})} \subset Y$ compact.

($K(\mathcal{U}) \subset Y$ relative compact.)

$\iff \quad \{u_n\} \subseteq X$ bounded sequence

$\Rightarrow \{Ku_n\}_{n \in \mathbb{N}}$ has a convergent subsequence in $Y$.

<u>Lemma</u> $\quad$ A compact operator is bounded.

<u>Proof</u> $\quad B_1(0) \subset X$ bdd $\Rightarrow K(B_1(0))$ cp. hence bdd in $Y$. $\quad K(B_1(0)) \subseteq B_R(0), \quad R > 0$.

$\Rightarrow \sup_{\|x\| \le 1} \|Kx\|_Y \le R \Rightarrow \|K\|_{op} \le R$.

qed

**Example 2** $A = -\frac{d^2}{dx^2}$, $\mathcal{D}(A) \to L^2[0,1]$ is unbounded, hence not compact.

**Example 1** $Kf(x) = \int_\Omega k(x,y) f(y)\,dy$, $K: L^2(\Omega) \to L^2(\Omega)$,

$k \in L^2(\Omega\times\Omega) \Rightarrow K$ is <u>compact</u>.

**Proof** R. p. 71, 72.

d) **Definition** $A: \mathcal{D}(A) \to H$ on a H-space. $A$ is <u>symmetric</u>, if

$$(Ax, y) = (x, Ay) \quad \forall\, x, y \in \mathcal{D}(A)$$

**Example 1** If $k(x,y) = k(y,x)$, then $K$ is symmetric on $L^2(\Omega)$.

$$(Kf, g) = \int\int k(x,y) f(y) g(x)\,dy\,dx$$

$$= \int\int f(y)\, k(y,x) g(x)\,dx\,dy$$

$$= (f, Kg).$$

**Example 2** $A = -\frac{d^2}{dx^2}$, $\mathcal{D}(A) := \{ f \in C^2[0,1],\ Af \in C^2[0,1],$
$\frac{df}{dx}(0) = 0,\ \frac{df}{dx}(1) = 0 \}$

$$(Au, v) = \int_0^1 -\frac{d^2}{dx^2} u(x) v(x)\,dx = \underbrace{-\frac{d}{dx}u \cdot v\Big|_0^1}_{=0} + \int_0^1 \frac{d}{dx}u \frac{d}{dx}v\,dx$$

$$= u\frac{d}{dx}v\Big|_0^1 - \int u \frac{d^2}{dx^2} v\,dx = (u, Av).$$

1.0

Theorem 1 (Hilbert-Schmidt) $A: H \to H$ linear, symmetric, compact, $H$ Hilbert-space. Then

(i) All e-values of $A$ are real and there is at most one accumulation point at $\lim_{n\to\infty} \lambda_n = 0$.

(ii) The e-vectors $(w_j)$ can be chosen to form an orthonormal basis and

$$Au = \sum_{j=1}^{\infty} \lambda_j (u, w_j)\, w_j$$

Theorem 2 (Spectral Theorem) $A: \mathcal{D}(A) \to H$, $R(A) = H$, symmetric linear unbounded. $A^{-1}$ exists and is compact. then

(i) exist an infinite set $\{\lambda_n\}$ of real e-values with $\lim_{n\to\infty} \lambda_n = +\infty$

(ii) The e-vectors $\{w_j\}$ can be chosen to form an orthonormal basis and

$$Au = \sum_{j=1}^{\infty} \lambda_j (u, w_j)\, w_j$$

e) Definition An operator $A$ on a $H$-space is positive, if there is a constant $k > 0$ such that

$$(Au, u) \geq k \|u\|^2, \qquad u \in \mathcal{D}(A).$$

In this case, all its e-values are positive.

**Example 1**: If $h(x,y) \geq \delta > 0$, then $K$ is positive on $L^2_+(\Omega)$

$$(Ku,u) = \iint h(x,y)\, u(x)\, u(y)\, dx\, dy$$

$$\geq \delta \iint u(x) u(y)\, dx\, dy \geq \delta \int \int_{\{y=x\}} u(x) u(y)\, dy\, dx$$

$$= \delta \|u\|_2^2 .$$

**Example 2** $A = -\frac{d^2}{dx^2}$, $\mathcal{D}(A)$ as above is <u>positive</u>.

$$(Au,u) = \int_0^1 -\frac{d^2}{dx^2} u(x)\, u(x)\, dx = + \int_0^1 \frac{d}{dx} u \frac{d}{dx} u\, dx$$

$$= \int_0^1 \left|\frac{d}{dx} u\right|^2 dx \geq \int_0^1 |u|^2 dx = \|u\|^2 .$$

Poincaré Inequality ($\to$ later).

## f) Fractional Powers

If a positive operator $A$ has a representation as

$$Au = \sum_{j=1}^{\infty} \lambda_j (u, w_j) w_j$$

then we define the <u>fractional powers</u> as

$$A^{\alpha} u = \sum_{j=1}^{\infty} \lambda_j^{\alpha} (u, w_j) w_j$$

for $\alpha \in \mathbb{R}$, $\alpha \geq 0$.

$$\mathcal{D}(A^\alpha) = \{u : \|A^\alpha u\| < \infty\}$$

$$= \left\{ u : u = \sum_{j=1}^{\infty} c_j w_j , \quad \sum_{j=1}^{\infty} |c_j|^{2\alpha} \lambda_j^2 < \infty \right\}$$

$\mathcal{D}(A^\alpha)$ is a H-space with inner product.

$$(u, v)_{\mathcal{D}(A^\alpha)} := (A^\alpha u, A^\alpha v)$$

norm $\quad \|u\|_{\mathcal{D}(A^\alpha)} = \|A^\alpha u\|$