(2.4) Hilbert-spaces, H

= complete vector space with inner product $(\cdot,\cdot)_H$.

natural norm $\|x\|_H = (x,x)_H^{\frac{1}{2}} \qquad x \in H$

$L^2(\Omega)$ is Hilbert-space with $(x,y)_{L^2} = \int_\Omega x(t)y(t)\,dt$

Definition a) $\{e_j\} \in H$ is orthonormal, if

$$x = \sum_{j=1}^{\infty} (x, e_j)\, e_j \qquad \forall x \in H$$

b) Let $\{e_j\}$ orthonormal. If in addition

$$\|x\|^2 = \sum_{j=1}^{\infty} (x, e_j)^2 \qquad \text{for all } x \in H,$$

then $\{e_j\}$ is called basis.

Proposition: H is separable, if it has a countable basis.