(2.3) Some useful integral estimates

Robinson p. 22 Thm. 1.7

Theorem 1 (Monotone Convergence)

$0 \le f_1(x) \le f_2(x) \le \dots$ a.e. $x \in \Omega$, f_j measurable

then $$\lim_{n\to\infty} \int_\Omega f_n(x)\,dx = \int_\Omega \left(\lim_{n\to\infty} f_n(x) \right) dx$$

Theorem 2 (Fatou's Lemma) $f_j \ge 0$, measurable:

then $$\int_\Omega \left(\liminf_{n\to\infty} f_n(x) \right) dx \le \liminf_{n\to\infty} \int_\Omega f_n(x)\,dx$$

Theorem 3 (Dominated Convergence)

$f_j \ge 0$, $f_j(x) \to f(x)$ for all $x \in \Omega$, and $|f_j(x)| \le g(x)$
$g \in L^1$ then
$$\lim_{n\to\infty} \int_\Omega f_n(x)\,dx = \int_\Omega \left(\lim_{n\to\infty} f_n(x) \right) dx .$$

Theorem 4 Young's inequality

If $a, b \ge 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (i.e. p and q are conjugate) then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

and if $\varepsilon > 0$:

$$ab \le \varepsilon a^p + \varepsilon^{-\frac{q}{p}} b^q$$

<u>Theorem 5</u> (Hölders inequality) (p,q) conjugate indices $1 \le p \le \infty$, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $f g \in L^1(\Omega)$ and

$$\| f \cdot g \|_1 \le \| f \|_p \, \| g \|_q$$

<u>Corollary</u> Ω bounded then $L^r(\Omega) \subset L^s(\Omega)$ for $r > s$.

<u>Proof</u>. Take $(p,q) = \left(\frac{r}{r-s}, \frac{r}{s} \right)$, then

$$\int_\Omega |f(x)|^s \, dx = \int 1 \cdot |f(x)|^s \, dx$$

$$\le \left(\int 1 \right)^{\frac{r-s}{r}} \left(\int |f(x)|^r \, dx \right)^{\frac{s}{r}}$$

$$= |\Omega|^{\frac{r-s}{r}} \, \| f \|_r^s$$

<u>Theorem 6</u> (Minkowski's inequality)

$$\| f + g \|_p \le \| f \|_p + \| g \|_p$$

<u>Theorem 7</u> $C_c^0(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$
$C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$
$L^p(\Omega)$ is separable.

Differential inequalities

One-sided derivative. $\frac{d}{dt_+}x(t) = \lim_{h\to 0^+} \frac{x(t+h)-x(t)}{h}$

if $x \in C^1(\mathbb{R})$ then $\frac{d}{dt_+}x(t) = \frac{d}{dt}x(t) \quad \forall t \in \mathbb{R}$.

Lemma

$$\frac{d}{dt_+}x \leq f(x,t), \qquad x(0) = x_0 \qquad (*)$$

$$\frac{d}{dt}y = f(y,t), \qquad y(0) = y_0 \qquad (**)$$

f Lipschitz continuous. If $x_0 \leq y_0$ then

$$\boxed{x(t) \leq y(t)} \qquad \text{on } [0,T].$$

Proof. Let $y_n(t)$ solve $\frac{dy_n}{dt} = f(y_n,t)+\frac{1}{n}$, $y_n(0) = y_0$

Claim $x(t) \leq y_n(t)$.

Assume not. Then $\exists\, t_1 > 0$: $x(t_1) > y_n(t_1)$

$\Rightarrow \exists\, t_2 \in [0,t_1)$: $x(t_2) = y_n(t_2)$. Choose t_2 to be the largest such value, such that

$x(t) > y_n(t) \quad \forall t \in (t_2,t_1)$.

Then $\frac{x(t_2+h)-x(t_2)}{h} > \frac{y_n(t_2+h)-x(t_2)}{h}$

$$\Rightarrow \frac{d}{dt_+}x(t_2) \geq f(y_n(t_2),t_2)+\frac{1}{n}$$

$$= f(x_n(t_2),t_2)+\frac{1}{n}$$

↯ contradiction to $(*)$

Now as $x(t) \le y_n(t)$ $\forall n \in \mathbb{N}$ we pass to the limit $n \to 0$. One can show that
$y_n(t) \longrightarrow y(t)$ solution of $\frac{d}{dt} y = f(y,t)$, $y(0) = y_0$.
Hence $x(t) \le y(t)$.

qed.

Gronwall's Lemma:

$$\frac{d}{dt_+} x \le g(t)x + h(t)$$

Then

$$\boxed{x(t) \le x(0) \exp\left(\int_0^t g(s)ds\right) + \int_0^t \exp\left(\int_s^t g(\tau)d\tau\right) h(s) ds}$$

Special case

$$\frac{d}{dt^+} x \le ax + b$$

$$\Longrightarrow \boxed{x(t) \le \left(x_0 + \frac{b}{a}\right) e^{at} - \frac{b}{a}}$$

Proof. The right hand side is the unique solution of

$$\frac{d}{dt} y = g(t)y + h(t), \quad y(0) = x_0.$$

Then apply the above Lemma.