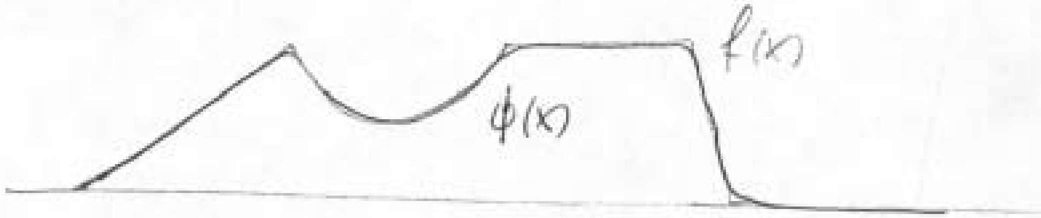
(2.2) Mollifiers

= Approximation by smooth functions:

Theorem: Given $f \in C_c^0(\Omega)$. Then for each $\varepsilon > 0$ there exists $\phi \in C_c^\infty(\Omega)$ such that

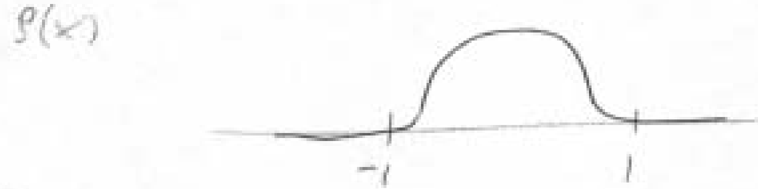
$$\| f - \phi \|_\infty < \varepsilon$$



Proof By explicite construction:

We define
$$\varrho(x) := \begin{cases} c \exp\left(\frac{1}{|x|^2 - 1}\right), & |x| \leq 1 \\ 0, & |x| \geq 1 \end{cases}$$

where c is chosen such that $\int_{\mathbb{R}^n} \varrho(x)\,dx = 1$.

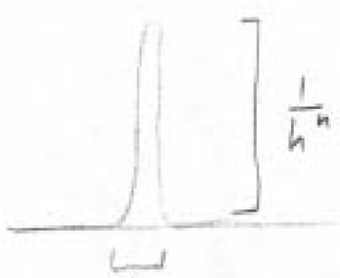


$\varrho \in C_c^\infty(\mathbb{R}^n)$

Standard mollifier of size ϱ

$$\varrho_h(z) := h^{-n} \varrho\left(\frac{z}{h}\right)$$

h small



$$\int_{\mathbb{R}^n} \varrho_h(x)\,dx = \int_{\mathbb{R}^n} \frac{1}{h^n} \varrho\left(\frac{x}{h}\right) dx \qquad z = \frac{x}{h}$$

$$dz = \frac{dx}{h^n}$$

$$= \int_{\mathbb{R}^n} \varrho(z)\,dz = 1. \qquad dx = h^n dz$$

given $f \in C^0_c(\Omega)$ then the <u>mollification</u> f_h of f, for $h < \operatorname{dist}(\operatorname{supp} f, \partial\Omega)$ is

$$f_h = f * \varrho_h$$

$$= \frac{1}{h^n} \int_\Omega \varrho\left(\frac{x-y}{h}\right) f(y)\,dy$$

<u>Proposition</u>. $f_h \in C^\infty_c(\Omega)$ and $f_h \to f$ as $h \to 0$ uniformly in Ω, in $C^0(\Omega)$.

<u>Proof</u>. $f_h = \frac{1}{h^n} \int_\Omega \varrho\left(\frac{x-y}{h}\right) f(y)\,dy \in C^\infty(\Omega)$, since $\varrho \in C^\infty$.

$\operatorname{supp} f_h \subset\subset \Omega \implies f_h \in C^\infty_c(\Omega)$.

note $\frac{1}{h^n} \int \varrho\left(\frac{x-y}{h}\right) dy = 1$, hence

$$|f_h(x) - f(x)| = \left| \frac{1}{h^n} \int \varrho\left(\frac{x-y}{h}\right) (f(y) - f(x))\,dy \right|$$

$$\leq \sup_{|x-y| \leq h} |f(y) - f(x)| \cdot \underbrace{\left| \frac{1}{h^n} \int \varrho\left(\frac{x-y}{h}\right) dy \right|}_{=1}$$

Since f is continuous we can make h small such that $|f(y)-f(x)| < \varepsilon \quad \forall |x-y| < h$.

$\Rightarrow |f_h(x) - f(x)| \leq \varepsilon \quad \forall x \in \Omega$.

qed