2 Some Functional Analysis

(2.1) Banach spaces

a) A Banach space $(X, \|\cdot\|_X)$ is a <u>complete normed</u> space.

<u>Example</u>: $\mathbb{R}^n$ with each norm: $x = (x_1, \dots, x_n)$

sum-norm: $\|x\|_1 = \sum_{j=1}^{n} |x_j|$

Euclidean-norm: $\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}$

p-norm: $p \geq 1$: $\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$

max-norm: $\|x\|_\infty = \max_{j=1,\dots,n} |x_j|$

b) A subset $Y \subset X$ is <u>dense</u>, if $\overline{Y} = X$.

A Banach space which contains a <u>dense countable</u> subset is called <u>separable</u>

<u>Ex</u>: $\mathbb{R}^n$ is separable, $\overline{\mathbb{Q}}^n = \mathbb{R}^n$.

d) A subset $E \subset X$ is called <u>compact</u>, if

(i) each open cover contains a finite subcover

$\Leftrightarrow$ (ii) each sequence $\{x_n\}_{n \in \mathbb{N}} \subset E$ has a convergent subsequence.

<u>Ex</u>. $\overline{B_1(0)} = \{x \in \mathbb{R}^n \; \|x\|_2 \leq 1\}$ is compact

(it is bounded and closed. Heine-Borel)

e) Two norms $\|\cdot\|^{(1)}$, $\|\cdot\|^{(2)}$ are equivalent, if there exist $a, b > 0$:

$$a \|x\|^{(1)} \leq \|x\|^{(2)} \leq b \|x\|^{(1)} \quad \forall x \in X.$$

Theorem: All norms on $\mathbb{R}^n$ are equivalent

Proof: In class presentation ② [→ R. p. 13, 14].

f) Spaces of continuous functions

$\Omega \subseteq \mathbb{R}^n$, closure $\bar{\Omega}$, boundary $\partial\Omega$.

$C^0(\Omega) := \{ f : \Omega \to \mathbb{R} : \text{continuous} \}$

$C^0(\bar{\Omega}) := \{ f : \bar{\Omega} \to \mathbb{R} : \text{continuous} \}$

$C^0_b(\Omega) := \{ f \in C^0(\Omega) : \text{bounded} \}$

norm: $\quad \|u\|_\infty = \sup\limits_{x \in \Omega} |u(x)|$

Prop: Ω bounded: $C^0_b(\Omega) = C^0(\Omega)$

$C^0(\bar{\Omega})$ is separable Banach-space

Derivatives $\quad D_j := \dfrac{\partial}{\partial x_j}$

$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $\quad \alpha = (\alpha_1, \dots, \alpha_n)$ multiindex

$C^r(\Omega) = \{ f : \Omega \to \mathbb{R} : D^\alpha f \in C^0(\Omega) \text{ for all } |\alpha| \leq r \}$

norm: $\quad \|f\|_{C^r} = \sum\limits_{|\alpha| \leq r} \sup\limits_{x \in \Omega} |D^\alpha f(x)|$

$$C^\infty(\bar\Omega) = \bigcap_{r=1}^{\infty} C^r(\bar\Omega)$$

<u>Support</u>. $\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$

$$C_c^r(\Omega) := \{ f : \operatorname{supp} f \text{ is compact} \}$$
$$= \{ f : \operatorname{supp} f \subset\subset X \}$$

$C_c^\infty(\bar\Omega)$ is <u>not</u> a Banach-space (in general)

<u>Hölder-spaces</u> $0 \le \gamma \le 1$

$$C^{r,\gamma}(\bar\Omega) = \{ f \in C^r(\bar\Omega) : |D^\alpha f(x) - D^\alpha f(y)| \le C|x-y|^\gamma$$
$$|\alpha| = r,\ x, y \in X \}$$

norm:

$$\|f\|_{k,\gamma} = \|f\|_{C^k} + \sup_{\substack{x,y \in \bar\Omega \\ |\alpha| = r}} \frac{D^\alpha f(x) - D^\alpha f(y)}{|x-y|^\gamma}$$

Ω bounded:

$$C^r(\bar\Omega) \subseteq C^{r,\gamma}(\bar\Omega) \subseteq C^{r+1}(\bar\Omega).$$

<u>L^p-spaces</u> $1 \le p < \infty$.

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{R}$$
$$\|f\|_p := \Big(\int_\Omega |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty \}$$

L^∞-space

$$\operatorname*{esssup}_{\Omega} |f(x)| = \inf \left\{ \sup_{x \in S} |f(x)| \cdot S \subset \bar{\Omega} \text{ with } \Omega \setminus S \text{ has measure zero} \right\}$$

example 1 $\quad f(x) = \begin{cases} 1 & \text{else} \\ 2 & \text{at } x = 0 \end{cases}$

has $\operatorname*{esssup}_{[-1,1]} |f(x)| = 1$.

norm $\quad \|f\|_\infty = \operatorname*{esssup}_{\Omega} |f(x)|$

example 2 $\quad$ If $f \in C^0(\Omega)$ then $\operatorname*{esssup}_{\Omega} |f(x)| = \sup_{\Omega} |f(x)|$

Proposition: Ω have finite volume $\left(\int_\Omega 1 \, dx < \infty \right)$, then

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$$

and if $\|f\|_p \le K \quad \forall p$, then $\|f\|_\infty \le K$.

L^p_{loc}-spaces $\quad \Omega$ unbounded.

$L^p_{loc}(\Omega) = \{ f : f \in L^p(K) \text{ for every } K \subset\subset \Omega \}$

ex: $f(x) \equiv 1$ on $\mathbb{R}$. $\quad \int_{\mathbb{R}} f(x) dx = +\infty \Rightarrow f \notin L^1(\mathbb{R})$

but $f \in L^1_{loc}(\mathbb{R})$. $\quad \int_K f(x) dx = |K| < \infty \quad \forall K \subset\subset \mathbb{R}$.